Artwork (cropped) from “Shots of Awe” YouTube series by Jason Silva
Jason Silva (Venezuelan Media Artist, Futurist, Philosopher, 1982 -)
http://thisisjasonsilva.com/

Infinite Algebra

> Our minds are finite, and yet even in these circumstances of finitude we are surrounded by possibilities that are infinite, and the purpose of life is to grasp as much as we can out of that infinitude.
>
> **Alfred North Whitehead** (English Mathematician and Philosopher; 1861 - 1947)

To many the infinite is a mysterious, troubling, metaphysical notion. The paradoxes in the previous chapter may have reinforced these views, suggesting there is little hope of grasping the infinite in any meaningful way. If you feel this way you are not alone - through history this view has been widely held.

Yet for many the infinite has served as the ultimate puzzle, the supreme challenge. A select group of nineteenth century mathematicians rose to this challenge and helped us put this puzzle together.

After exploring some of this trailblazing work, we shall see that despite being finite beings with finite experiences, we can both understand and use the infinite in meaningful, logical, and mathematically rigorous ways.

Driving Question - 0.999…

What can you say about the infinitely repeating decimal 0.999...?

Think about the driving question. Talk with some peers about this question. Make some conjectures. Write down some questions about 0.999... that it may be important to answer. See what progress you can make in understanding this infinite mathematical object.

Algebra and Infinitely Repeated Decimals

With the the Driving Question in mind, let's proceed using "simple" algebra and arithmetic to analyze the infinitely repeated decimal 0.121212....[1] We're not sure what this number is. Often we denote an unknown quantity by a variable so we are free to work with it. So define this quantity by the variable x, i.e.

$$x = 0.121212...$$

Multiplying both sides of this equation by 100 yields

$$100x = 12.1212...$$

This seems to have accomplished little more than creating another number, 10x, with the same unpleasant infinite string of decimal digits. However, employed creatively, this scaling can help us. Consider the following difference:

$$\begin{array}{r} 100x = 12.121212\ldots \\ -x = \ 0.121212\ldots \\ \hline \end{array}$$

Think about carrying out this subtraction. All of the decimal digits on the right, all infinitely many of them, cancel out those on the left leaving only the number 12 as a result. But the left hand side is a simple algebraic expression, leaving, upon subtraction, 99x. So the difference, once simplified, becomes

$$99x = 12.$$

Solving for x we see that, $x = \frac{12}{99}$.

What have we done? We have converted an infinite object into a finite one, a simple fraction.

$$0.121212\ldots = \frac{12}{99}$$

If you want, you can check this on your calculator for some reassurance.

[1] The notation 0.12 is often used instead of 0.121212... We will employ the latter notion because the ellipsis ...can be used in many different settings involving the infinite.

Investigation

1. Describe any thoughts, ideas, questions, and/or conjectures that you made about 0.999... when you thought about the Driving Question on page 2.

2. Define x = 0.999... Scale this number by a factor of 10. That is, determine the value of $10x$.

3. Can you adapt the method used above to with 0.121212... to determine an alternative, non-decimal identity for 0.999...? Explain.

4. Is your result in Investigation 3 intuitively satisfying to you? Explain. The result in Investigation 3 is often surprising to people. So let's investigate the number 0.999... in an alternative way.

5. What is the exact value of $\frac{1}{3}$ as a decimal?

6. Check the previous result using long division to precisely write as a (possibly infinite) decimal. Express your result as an equation $\frac{1}{3} =$.

7. Multiply both sides of your equation from Investigation 6 by 3. What non-decimal identity does this suggest for the number 0.999...?

8. Does this additional analysis help your intuition?

Calculator Warning:
Here and below you should only use calculators to check finite calculations. As you have just seen the number 0.333... is an infinite process as a decimal, but a finite one as a fraction. Calculators are generally decimal based. This illustration is part of a more general principle - computers cannot deal with infinite processes, but the human mind can.

More Infinite Decimals

9. Consider the the number represented by the infinitely repeated decimal x = 0.373737.... Scale this number by a factor of 10; i.e. compute $10x$. Does this scaling help you to employ the method used above with 0.121212... to determine an alternative, non-decimal identity for 0.373737...? Explain.

10. Compute 100x. Does this scaling allow you to employ the method above to determine an alternative, non-decimal identity for 0.373737...? Explain.

11. Now consider the infinitely repeated decimal number x = 0.295295295.... Compute 10x. Does this scaling help you to employ the method used above to determine an alternative, non-decimal identity for 0.295295295...? Explain.

12. Compute 100x. Does this scaling allow you to employ the method above to determine an alternative, non-decimal identity for 0.295295295...? Explain.

13. Determine an alternative, non-decimal identity for 0.295295295....

14. Yet another infinitely repeated decimal number is 0.821982198219.... Find an alternative, non-decimal identity for this number.

15. Repeat Investigation 14 for the infinitely repeated decimal x = 0.123456123456123456....

16. Repeat Investigation 14 for the infinitely repeated decima l x = 0.123456789123456789123456789....

17. These examples should suggest a general method, or algorithm, for determining alternative, non-decimal identities of any infinitely repeating decimal numbers. Explain.

Infinite Objects - Numerically

Let's return to 0.999... for a moment and think about it numerically, considering the numbers as numbers on the number line.

18. Which number is closer to 1, 0.9 or 0.999...? Explain.

19. What is the distance between 1 and 0.9?

20. Which number is closer to 1, 0.99 or 0.999...? Explain

21. What is the distance between 1 and 0.99?

22. Which number is closer to 1, 0.999 or 0.999...? Explain

23. What is the distance between 1 and 0.999?

24. You should see a pattern forming. Describe this pattern precisely.

25. So how close to 1 is 0.999...? Can you conclude anything from this? Explain. What about infinite series?

Infinite Series

An infinite series is formed by adding infinitely many terms together. A simple example of an infinite series is

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$$

Here the ellipsis means that the terms in this series continue indefinitely, following the evident pattern. What does one do with a series? Series involve adding. It is natural to look for the sum. What is the sum of an infinite series? One's intuition might suggest that because this sum grows as each successive term is added it would grow without bound. That is, the series would be infinite not simply in the number of terms added, but that the sum of this series must be infinite as well. Such intuition, however, is wrong.

26. What are the next ten terms following $\frac{1}{8}$ in the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$?

27. Write the <u>sum of the first two terms</u> in the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$, i.e. $1 + \frac{1}{2}$, as a single fraction.

28. Repeat Investigation 27 for the <u>first three terms</u>.

29. Repeat Investigation 27 for the <u>first five terms</u>.

30. Repeat Investigation 27 for the <u>first nine terms</u>.

31. Do you see a pattern in your answers?

32. Use this pattern to predict the sum of the first 15 terms.

33. Do these calculations suggest a exact value for the sum of this infinite series? Explain.

At this point some people protest, “All this shows is that the one object gets closer and closer to another, getting infinitely close together. This doesn’t have to mean they are the same.” This is a legitimate objection. To answer the objection, we need to have an agreed upon definition of the numbers we are using: fractions, decimals, and even infinitely repeating decimals. These numbers are part of the real numbers which comprise the number line that you have drawn since elementary school.

The pioneering work of several mathematicians, particularly Augustin Louis Cauchy (French Mathematician; 1789 - 1857) and Karl Weierstrass (German Mathematician; 1815 - 1897), during the mid- to late-nineteenth century in a period which has become known as the Age of Analysis, was responsible for the definitions of the real numbers.

Infinite Objects - Algebraically

Another way to try to determine the sum of an infinite series is to proceed algebraically as we did above. Namely, we would like a way to scale our infinite series so we could compare it to the original in a way leads only to finite objects that we can analyze in the standard way. Consider again the sum

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$$

The key insight to scaling this infinite object is that each term in this series is a factor of as large as the one that precedes it.

34. Scale S by a factor of $\frac{1}{2}$; i.e. write $\frac{1}{2}S$ as an infinite series.

35. Compare S and $\frac{1}{2}S$. Is it possible to adapt the method we used with decimals so all but finitely many terms can be canceled? Explain.

36. Determine an exact value for S. How does it compare with your answer in Investigation 33?

Consider now the infinite series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \ldots$.

37. What are the next ten terms in this series?

38. Use a calculator to determine the value of the sum of the first 10 terms in this infinite series. Does this calculation suggest a value for the sum of this infinite series?

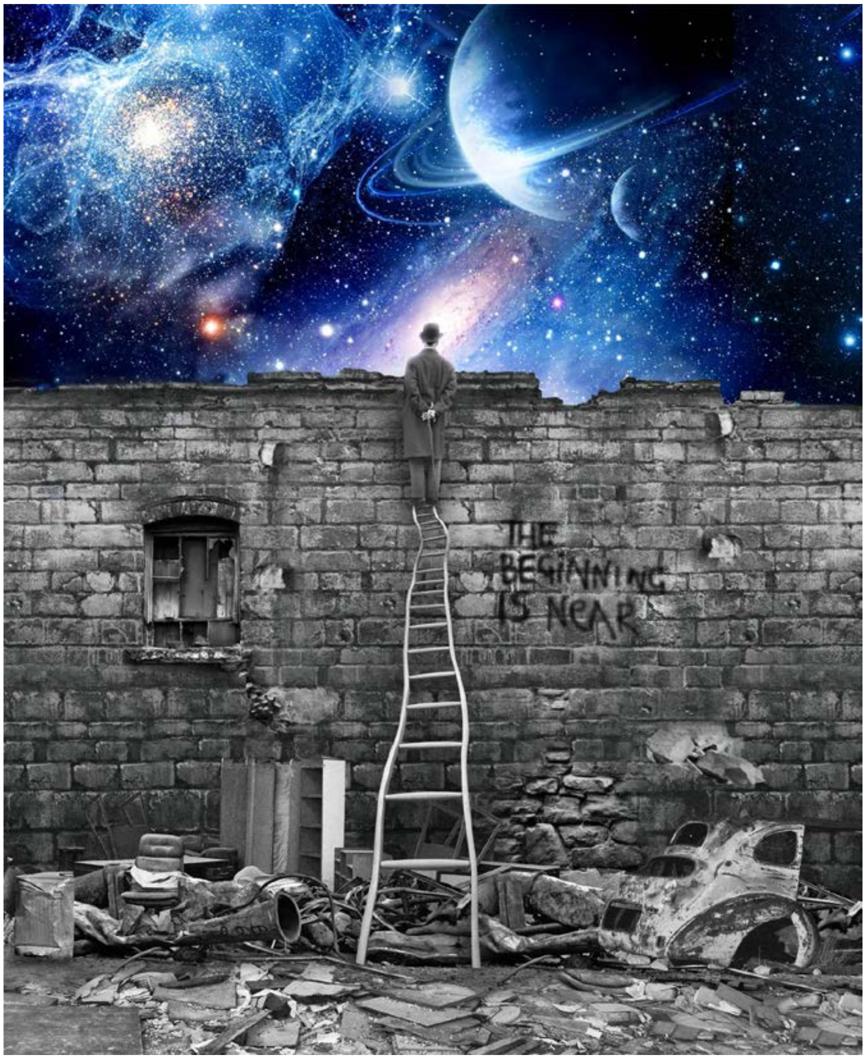
39. $\frac{1}{3}$ plays a critical role in relationship between consecutive terms in this infinite series. What role is this?

Denote the sum of this new series by $S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \ldots$.

40. Scale S by $\frac{1}{3}$; i.e. write $\frac{1}{3}S$ as an infinite series.

41. Compare S and $\frac{1}{3}S$.

42. Determine a value for S. How does your answer compare with your answer in Investigation Investigation 38?



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video on : http://paradigmmagazine.com/2012/02/20/paradigm-magazine-jason-silva-interview/
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