Sizes of Infinity

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not - I do not say divisible - but actually divisible; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.
>
> **Georg Cantor** (German mathematician;1845 - 1918)

> No one will expel us from the paradise that Cantor has created.
>
> **David Hilbert** (German mathematician;1862 - 1943)

Perspective Drawing

Our intuition suggests one Infinite; one definitive, universal, unlimited Infinite. As such Galileo's warning that "we cannot speak of infinite quantities as being the one greater or less than another," may seem entirely reasonable - infinity can have only one "size."

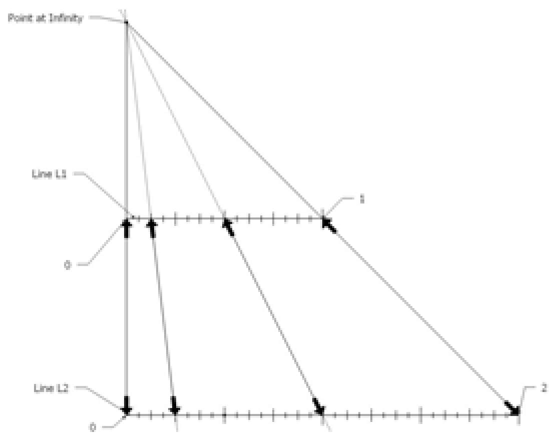
Nonetheless, it may have been suprising to see, during your exploration of the Wheel of Aristotle, that the number of points on a large circle is the same as the number of points on a small circle. We matched the points - all infinitely many of them - in a one-to-one way.

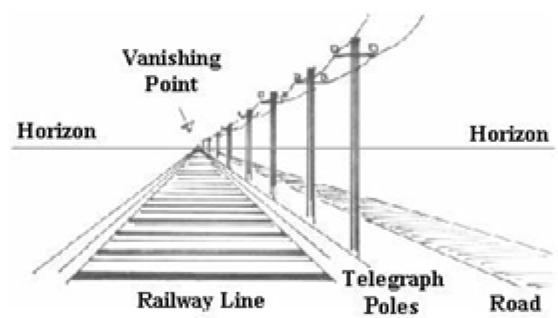
The discovery of perspective drawing was an important development in the history of art. It was also important mathematically. As Rudy Rucker (American mathematician,computer scientist, science fiction author, and philosopher;1946 -) tells us:

> Intellectually, perspective [drawing] is a breakthrough, because here, for the first time, the physical space we live in is being depicted as if it were an abstract, mathematical

> space. A less obvious innovation due to perspective is that here, for the first time, people are actually drawing pictures of infinities.[1]

Perspective drawing is a valuable tool in our efforts to compare different sizes of infinity. The figures below-right shows a basic one-point perspective drawing. Notice that all lines parallel to the line of sight of the viewer converge to the vanishing point which represents infinity. The other figure shows how a point at infinity allows us to match, i.e. put in a one-to-one correspondence, the points on a line of length one with the points in a line of length two. The number of points on each is the same.





Counting by Matching

Without question, the most important contributions to human understanding of the infinite were made by Georg Cantor (German mathematician;1845 - 1918). His work as a champion of the infinite brought him great personal joy coupled with extreme professional hostility and devastating emotional grief.

Cantor’s idea, and the profound revelations that follow from it, was simple. If we consider pursuing a quantitative study of the infinite, we need a new way to “count.” In the realm of the infinite our standard methods will be of little use. Cantor’s idea was to use matching as our way of counting.
It is an obvious choice. Those who cannot count, very small children for example, are certainly aware of such matching. They can tell you whether the number of popsicles in the freezer outnumber the children at the party, or conversely, without counting - they simply try to match popsicles to friends.

Since you already know how to count, let’s consider a few examples where matching is important.

[1] From Mind Tools: The Five Levels of Mathematical Reality.

1. The NCAA Women's Division 1 Basketball Tournament is a single elimination tournament with 64 teams. Determine how many games are played in this tournament.

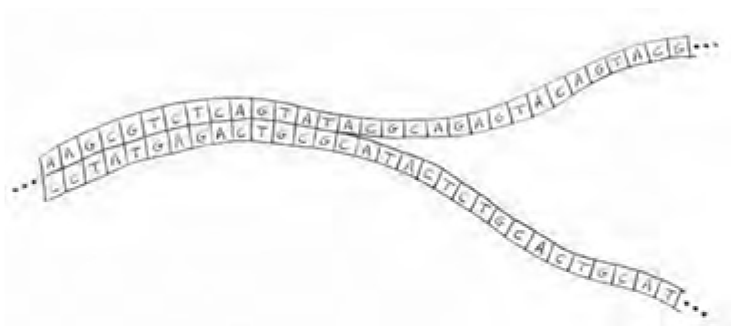
2. There are many ways to solve the problem in Investigation 1. A particularly elegant way uses matching. Match losers to games to show how you can immediately determine how many games are played in the tournament.

3. How many games are played in a single elimination tournament with 2^n teams? Explain.

4. Draw several polygons.[2] For each count the number of vertices, aka corners, and edges.

5. How does the number of vertices seem to be related to the number of edges in your polygons.

6. Devise a nice way to match vertices and edges that proves why your relationship in Investigation 5 holds for all polygons.



Strands of DNA

The figure above is a schematic diagram of Deoxyribonucleic acid, also known as DNA. Notice that there are two strands that come together like the two sides of zipper. Each letter represents one of the bases adenine(A), cytosine(C), guanine(G), and thymine(T). A base on one strand binds to the base opposite on the other strand. These are called DNA base pairs.

7. Look at the base pairs that make up the length of DNA in Figure 5.2. What do you notice about these pairs?

In fact, the matching that you describe in Investigation 7 is the only kind of matching that is possible, the so-called complementary base pairing. When organism grow DNA is replicated. Tremendously long lengths of DNA which are twisted and knotted up somehow knows how to

2 It is important to remember that polygons are simple, that is, their edges cannot cross and their is only one interior region.

unknot itself[3], zips into two pieces, and then matches free base pairs to each strand to complete the replication.

In other words, the very basis of life involves matching at the most basic level.

8. Think up your own non-trivial example of matching.

Now that you've thought about matching a bit, let's return to counting. What is it about the number three that gives it its "threeness"? What is common about three kids, three pebbles, three Magi, three daily meals, three colors on a stoplight, three races in horseracing's Triple Crown, and
all other things we say there are three of? What they share is that we can match the elements that make up each group (set) in a one-to-one way:

Addie ↔ granite ↔ Melchior ↔ breakfast ↔ red ↔
Kentucky Derby
Jacob ↔ quartz ↔ Caspar ↔ lunch ↔ blue ↔
Preakness stakes
KC ↔ feldspar ↔ Balthasar ↔ dinner ↔ green ↔
Belmont stakes

There are three in each group because each group can be matched with {1,2,3} which serves as the defining three element group.

To make this formal in a mathematical way we simply use the language that we have developed before. We notice that the objects of our study will be elements that we have grouped together as sets - just like in Chapter 2. Our way of matching is a **one-to-one correspondence** between two sets A and B which we define to be a rule[4] which matches each element from A with exactly one element of B and each element of B with exactly one element of A

Previously we used an intuitive notion of the size of sets - we just counted to find the cardinality of a given set. Now we can make a definition - one that we can use for the infinite as well. The set {a,e,i,o,u} has **cardinality** 5, by definition, because it can be put in a one-to-one correspondence with the set of the first five natural numbers, {1,2,3,4,5}. Namely, the matching rule that provides the one-to-one correspondence is

$$a \leftrightarrow 1 \quad e \leftrightarrow 2 \quad i \leftrightarrow 3 \quad o \leftrightarrow 4 \quad u \leftrightarrow 5.$$

Of course, not all sets have the same cardinality. How does our matching help us count here?

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[3] Actually the field of knot theory is playing a critical role in understanding the physical chemistry behind DNA replication

[4] More formally, the "rule" is a *function* that is both *one-to-one* and *onto*.

Here's an example. The cardinality of {a,e}, which is 2, is strictly less than the cardinality of {1,2,3,4,5}, which is 5, simply because

- a and e can be put in a one-to-one correspondence with any pair of elements from {1,2,3,4,5}
- a and e can never be put in one-to-one correspondence with all of {1,2,3,4,5}.

So we say the cardinality of {a,e} is strictly less than the cardinality of {1,2,3,4,5} and we write 2 < 5.

This is obvious, there's nothing surprising here. Yet...

Let us see what happens when we compare infinite sets in this way.

Comparing Infinite Sets

9. Let *S* = {1,4,9,16,...} be the set of squares. Name seven other elements of this set.

10. Intuitively, which is larger, the set S of squares or the set *N* = {1,2,3,...} of natural numbers?

11. Express the set *S* by writing each element as a square.

12. Use Investigation 11 to find a one-to-one correspondence between the set S of squares and the set *N* of natural numbers.

13. What does Investigation 12 tell you about the relative sizes of the set S of squares and the set N of natural numbers?

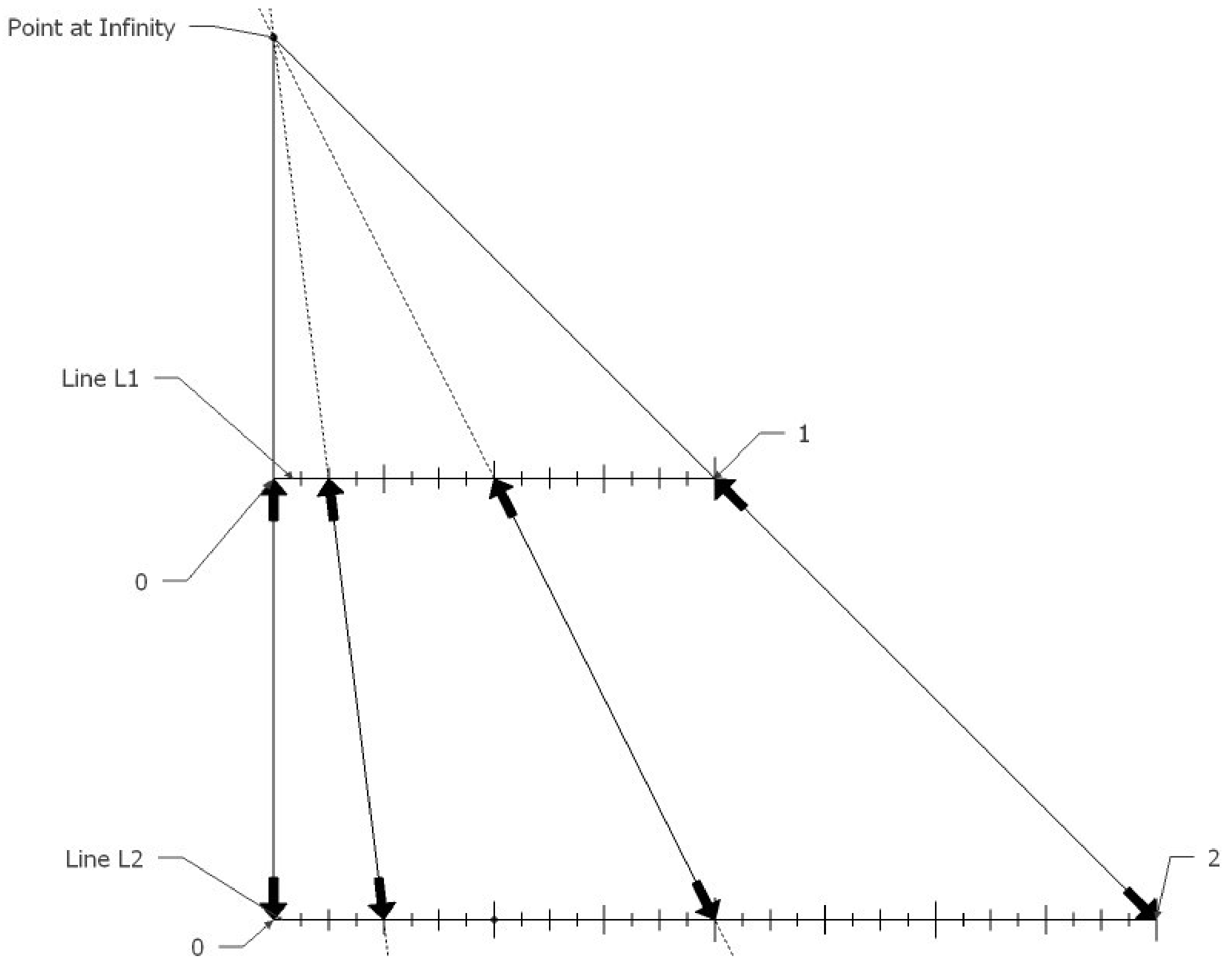
14. How does your answer to Investigation 13 compare with your answer to Investigation 10?

The apparent paradox indicated in Investigation 14is known as **Galileo's paradox**. This paradox prompted Galileo to conclude:

> We can only infer that the totality of all numbers is infinite, and that the number of squares is infinite...; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally, the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite quantities.

As we have noted, Galileo's conclusion is intuitively compelling and was generally accepted by mathematicians and philosophers for millennia. What we will discover is that if we open our minds as Cantor did, the infinite is a much richer, varied landscape.

The figure below shows a perspective-like drawing. The point 0 on L_1 is matched with the point 0 on L_2 and the point 1 on L_1 is matched with the point 2 on L_2.



15. Using the matching illustrated in the Figure above, to which point on L_2 does the point $\frac{1}{2}$ on L_1 correspond?

16. Similarly, to which point on L_1 would the point $\frac{1}{4}$ correspond to?

17. To which point on L_2 does the point $\frac{7}{8}$ on L_1 correspond to?

18. To which point on L_1 does the point $\frac{11}{16}$ on L_2 correspond to?

19. Explicitly find the correspondence between several other points on L_1 and L_2.

20. Can you find a general rule or description to describe the one-to-one correspondence between the points on L_1 and L_2 precisely?
21. Construct a new Figure to find a one-to-one correspondence between lines L_1 of length one and L_3 of length 3.

22. Can you describe this one-to-one correspondence precisely as you did the other in Investigation 20?

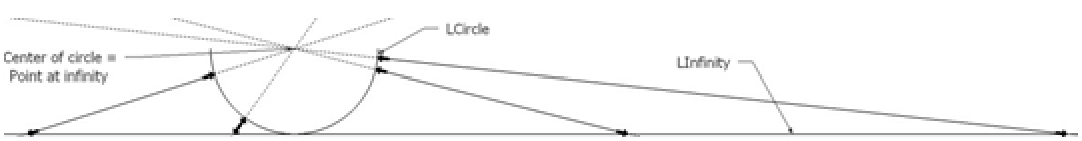
23. Construct another Figure to find a one-to-one correspondence between lines L_1 of length one and L_{10} of length 10.

24. Can you describe this one-to-one correspondence precisely as you did the other in Investigation 22?

25. Will the method you have been using provide a one-to-one correspondence between any two lines with finite lengths? Explain.

26. Can you use this method to find a one-to-one correspondence between a line L_1 of length one and a line L_∞ which extends indefinitely in both directions? Explain.

Consider the Figure below, where points on the semicircle *LCircle* are projected onto the line L_∞ extending indefinitely in both directions. In three dimensions, projections of this sort are the fundamental tool used to map our spherical earth onto flat maps. All projections introduce distortions, the type (e.g. distorted areas, distorted distances, or distorted angles) of which depends on the mathematical nature of the projection chosen. Which map is "best" is a continuing source of significant controversy.



Stereographic projection.

27. Does the Figure above help provide a one-to-one correspondence between L_C and L_∞? Explain.

28. What do all the results in Investigations 20-27 tell you about the cardinalities of line segments?