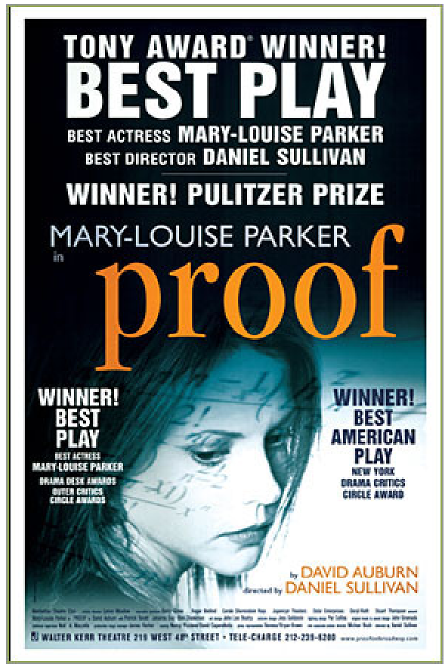
Proof



A good proof is one that makes us wiser.

Yuri I. Manin
(Russian mathematician; 1937 -)

In the previous unit, you investigated how several different systems were built up from axioms, providing a solid foundation for deductive conclusions. Beginning with the foundation often necessitates a long, steep climb.

One of the most important contributions to the *foundations of mathematics* is the massive, three-volume Principia Mathematica by **Bertrand Russell** (British mathematician, philosopher, and author; 1872 - 1970) and **Alfred North Whitehead** (English mathematician and philosopher; 1861 - 1947) which sought to formalize mathematics from a single set of axioms and deduce results using the rules of symbolic logic.

One of the results is the first fully *formalized* proof that 1+1 = 2. Using several earlier results from the 680 page Volume I, this proof is completed on p. 68 of Volume II!

***110·643.** $\vdash . 1 +_c 1 = 2$

Dem.

$$\vdash . *110{\cdot}632 . *101{\cdot}21{\cdot}28 . \supset$$
$$\vdash . 1 +_c 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota' y \in 1\}$$
$$[*54{\cdot}3] \quad = 2 . \supset \vdash . \text{Prop}$$

The above proposition is occasionally useful. It is used at least three times, in *113·66 and *120·123·472.

Last step in Russell and Whitehead's formal proof that 1 + 1 = 2

Although significant work on the foundations of mathematics continues, the day to day work of most mathematicians relies on proofs based on well accepted structures. For example, a well-known principle like the *distributive law* for integers is used regularly without concern.

In other words, up in the tree of mathematics, high above the foundations, mathematicians work building off of the results of others. As these previous results have been established deductively, mathematicians know that if they proceed from these established results their work will be valid as well. The role of deductive reasoning remains a *sine qua non* - essential ingredient - of the mathematical process. Only here, to signify that we are moving from known results to new results established logically, the words *proof* and *prove* are typically used to describe logical argument and logical process.

Number Theoretic Proofs

1. Explain what even and odd numbers are.

2. We often explain things intuitively. If you had to give a rigorous definition of even numbers, what would it be? What about odd numbers? Explain.

The typical definition of an **even number** is:

- A positive integer is even if it can be written as 2n where n is some non-negative integer.

The definition of an **odd number** is analogous:

- A positive integer is odd if it can be written as 2n + 1 where n is some non-negative integer.

3. Are your definitions in 2 equivalent to those just given? If so, prove your result. If not, provide an example which illustrates the difference.

4. Take several pairs of odd counting numbers and multiply each pair together. What do you notice about the products of these pairs of odd counting numbers?

5. Using the pattern you have observed in 4, state a conjecture that characterizes the product of any two odd counting numbers.

Here we demonstrate how this result can be proven deductively:

Proof. Denote the two counting numbers by a and b. By assumption, both a and b are odd. By definition this means that there are positive integers n and m so that $a = 2n + 1$ and $b = 2m + 1$. Then the product $a \cdot b$ is given by:

$$\begin{aligned} a \cdot b &= (2n+1)\cdot(2m+1) \\ &= 4nm + 2n + 2m + 1 \\ &= 2\,(2nm + n + m)) + 1 \end{aligned}$$

$2nm + n + m$ is a positive integer and so, by definition, $a \cdot b$ is odd.

6. Take several pairs of even counting numbers and add each pair together. What do you notice about the sums of these pairs of even counting numbers?

7. Using the pattern you have observed in 10 state a conjecture that characterizes the sum of any two even counting numbers.

8. Using the definition of even numbers, prove your conjecture about the sum of two even numbers.

9. Do you see a way to prove your conjecture using your definition in 2? Explain.

10. Take several pairs of odd counting numbers and add each pair together. What do you notice about the sums of these pairs of odd counting numbers?

11. Using the pattern you have observed in 10 state a conjecture that characterizes the sum of any two odd counting numbers.

12. Prove your conjecture about the sum of two odd numbers.

13. Take several pairs of counting numbers, one even and one odd, and multiply each pair together. What do you notice about the products of these pairs, one even and one odd, of counting numbers?

14. Using the pattern you have observed in 13 state a conjecture that characterizes the product of any two, one even and one odd, counting numbers.

15. Prove your conjecture about the product of an even and an odd number.

16. Prove that the square of an even number is even.

17. Prove that the square of an odd number is odd.