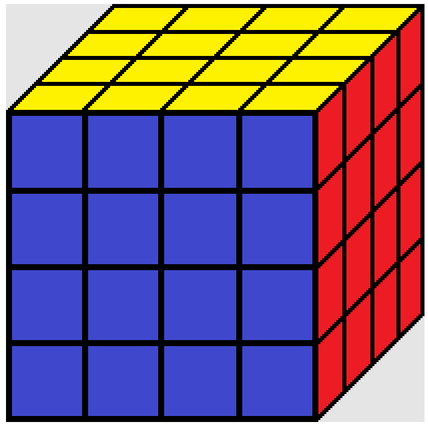

Magical Cube Moves

There are many ways of solving the cube. Some are very fast but may require you to memorize (or look up) many different specialized moves (see, for example, www.speedcubing.com). Other methods succeed with a smaller set of moves, but may take longer, or require you to start again from the very beginning when something goes wrong. Some methods start by solving the first layer. And, it has now been proven, that God can solve the cube, no matter what state it is in, in at most 20 moves.[1] Since we are mortal, we're going to begin slow by analyzing some very specific moves that are fairly magical. These will lead us, in the next section, to strategies for solving the cube.
The first magical move we will consider, which we'll label by M_1, is:

$$M_1 = R^{-1}DRD^{-1}$$

The algebraic structure of this move certifies it as what is called a *commutator*. Indeed, the cube offers a wonderful *microworld* for the exploration of what is known as *modern abstract algebra*. The index of Adventures in Group Theory by **David Joyner** is a who's who of famous mathematicians and fundamental topics in modern abstract algebra - the body of the book suitable for an advanced undergraduate course for mathematics majors.

To begin experimenting with M_1, solve the top layer of the cube.

1. Beginning with the top layer of the cube solved, perform the move M_1. What impact did this move have on the top layer of the cube?

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[1] The minimum number of moves needed to solve an arbitrary cube has been an open question, drawing a great deal of attention for many years. This number is known as **God's number**. The first positive result was that at least 18 moves were necessary and 52 moves suffice, proven by **Morwen Thistlethwaite** (; -) in 1981. In 1995 it was shown that the "superflip" required 20 moves, but the upper bound remained at 29 moves. It stood at this point for 10 years. The great breakthrough came in August, 2010. With the help of lots of computer CPU time at Google - Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge proved that 20 is both necessary and sufficient. See http://www.cube20.org/ for more information.

2. Perform M_1 again. Did it bring the top layer back? If not, what did it do?

3. Keep performing M_1 over and over again until the top layer is restored to its solved state. How many moves did it take you? Can you explain geometrically why it took you this number of moves?

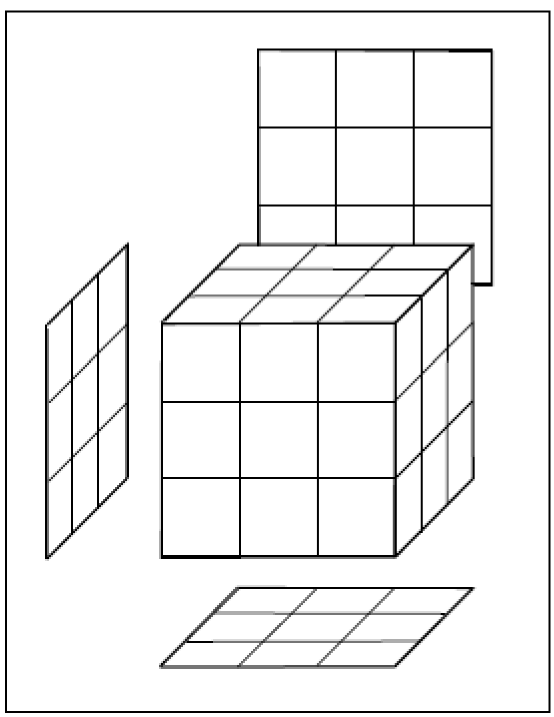
We would like to understand not only what the move M_1 does to the cubies on the top layer of the cube, but to all cubies.

4. There are a number of cubies that are not moved by any of the moves that make up M_1. Describe exactly which cubies these are.

5. There are several cubies that are moved by the individual moves that make up M_1 but nonetheless, when M_1 is completed, these cubies return to their original location with their original orientation. Describe exactly which cubies these are. (We strongly suggest using a friend to help.)

6. There are several cubies who locations and orientations are changed by M_1. Describe the impact of M_1 on these cubies by:

- Drawing arrows on the cube in the Figure belowto show were moved cubies move to, and,
- Indicating precisely which cubies get interchanged with their names.



7. Now that you know the exact impact of M_1 on the cube, can you predict how many times in a row you will have to perform this move before the cube is brought back to the configuration you started from for the first time? Explain.

8. Check to see that your prediction is correct.

The number you found in Investigation **8** is called the *order* of M_1.

Now we'll look at another magical move, which we will call M_2. It involves not only the usual moves, but a new move which you may have already used: M_R. M_R is a clockwise rotation of the middle slice parallel to the right side (clockwise as seen from the right side). Our new magic move is:

$$M_2 = M_R U M_R U M_R U M_R U.$$

Again, to begin, solve the top layer of the cube.

9. Can you think of a shorthand, algebraic way to write M_2?

10. Beginning with the top layer of the cube solved, perform the move M_2. What impact did this move have on the top layer of the cube?

11. Keep performing M_2 over and over again until the top layer is restored to its solved state for the first time. How many moves did it take you? Can you explain geometrically why it took you this number of moves?

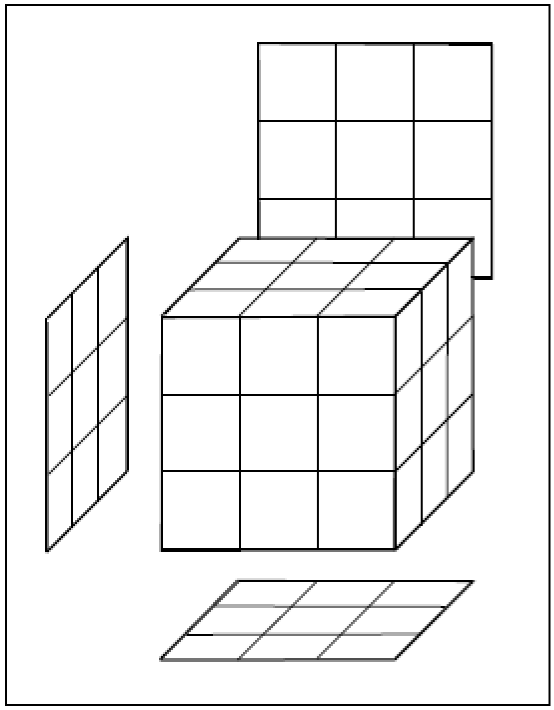
We would like to understand not only what the move M_2 does to the cubies on the top layer of the cube, but to all cubies.

12. There are a number of cubies that are not moved by any of the moves that make up M_2. Describe exactly which cubies these are.

13. There are several cubies that are moved by the individual moves that make up M_2 but nonetheless, when M_2 is completed, the cubies return to their original location with their original orientation. Describe exactly which cubies these are. (We strongly suggest using a friend to help.)

14. There are several cubies who locations and orientations are changed by M_2. Describe the impact of M_2 on these cubies by:

- Drawing arrows on the cube in the Figure below to show where moved cubies move to, and,
- Indicating precisely which cubies get interchanged with their names.



15. Now that you know the exact impact of M_2 on the cube, can you predict what the

16. Check to see that your prediction for the order of the move M_2 is correct.

17. We called M_1 and M_2 magical moves. We did that because they only move a small number of cubies. Why might this be useful when we are solving the cube?

18. If our goal is to find moves whose net results are to move a very small number of cubies, do you think the number of basic moves that are used in these moves are going to be small or large? Explain.