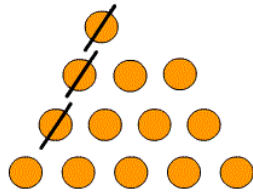
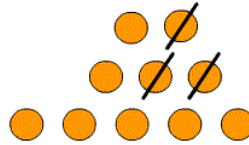


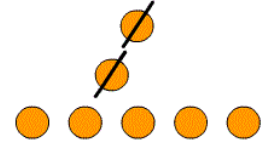
First player



Second player



First player



Second player

## One pile, two pile, three piles

### One pile Nim

**Rules:** One pile Nim is a two-player game. Place a small handful of stones in the middle. At every turn, a player decides whether to take one, two, or three stones from the pile. Whoever takes the last stone(s) wins.

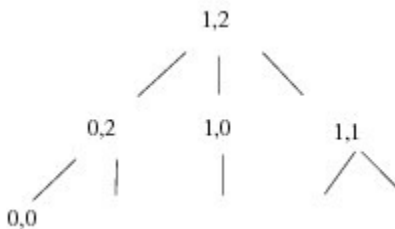
1. Play the game a few times with a partner.
2. At what point of the game do you know if you've lost or won? Explain why. How many stones were left in the pile when you knew that?
3. Can you find a higher number of stones so that you're sure you will win or lose?
4. How about the next highest number of stones? Is there a pattern? (How high can you go?)
5. Suppose there are 4 stones in the pile. Do you want to go first or second? Explain why.
6. Suppose there are 9 stones in the pile. Do you want to go first or second? Explain why.
7. Suppose there are 15 stones in the pile. Do you want to go first or second? Explain why.
8. Explain the full strategy of the game (a "strategy" tells you precisely what to do in any situation).
9. Now that you know the strategy, is this game still interesting to play?
10. Now we change the rules of the game so you can remove up to five stones at each turn. What is your strategy now?
11. Now the players are allowed to remove up to thirteen stones at each turn? What is your strategy now?

## Two pile Nim

**Rules:** Now, we place *two* piles of stones in the middle. Players take turns. At each turn, you get to pick a pile and take as many stones as you want from that pile. Whoever takes the very last stone(s) wins.

**Example:** For example, I could take one whole pile. But then my opponent could take the other pile and I would lose.

12. Play the game a few times with a partner.
13. At what point of the game do you know if you've lost or won? Explain why.
14. Can you find a situation (i.e. two piles with a certain number of stones each) so that you're sure that you will lose? Explain why.
15. Can you find a situation with higher numbers of stones so that you're sure that you will lose? Explain why.
16. Suppose there are 2 stones in each pile. Do you want to go first or second? Explain why.
17. If you feel that you have a good strategy, try the following two situations. If not, take a look at Investigation **20** and come back later.
  - a. Suppose there are 7 and 9 stones in the respective piles. Do you want to go first or second? Explain why.
  - b. Suppose there are 6 and 25 stones in the respective piles. Do you want to go first or second? Explain why.
18. Explain the full strategy of the game (a strategy tells you precise what to do in any situation).
19. Now that you know the strategy, is this game still interesting to play?



20. Consider the situation in the tree diagram above, with 2 stones in one pile, and one stone in the other. Anna plays against Ben. Anna gets to go first. The following tree lists all of Anna's possibilities for her first move, and one scenario where Ben ends up winning. Fill in the missing scenarios.

21. Which move should Anna start with in order to win? Explain.
22. Use a similar tree to analyze the game starting with two stones in both piles.
23. Are there any subtrees that you have already seen before? Which ones?
- 24. Classroom Discussion:** What precisely do we mean when we say “Anna has a winning strategy?” How can you use winning strategies to create shortcuts in your tree notation?
25. Next, analyze the game tree for a game with two stones in one pile and three in the other, i.e. starting with 2,3. Use shortcuts whenever possible.
26. Analyze the 3,3 game using a tree.
- 27. Classroom Discussion:** Bringing together our understanding from Investigations 17a-17b and our game tree investigations, what is a winning strategy for “Two Piles”?

## Three pile Nim

**Rules:** Now, we place *three* piles of stones in the middle. Players take turns. At each turn, you get to pick a pile and takes as many stones as you want from that pile. Whoever takes the last stone(s) wins.

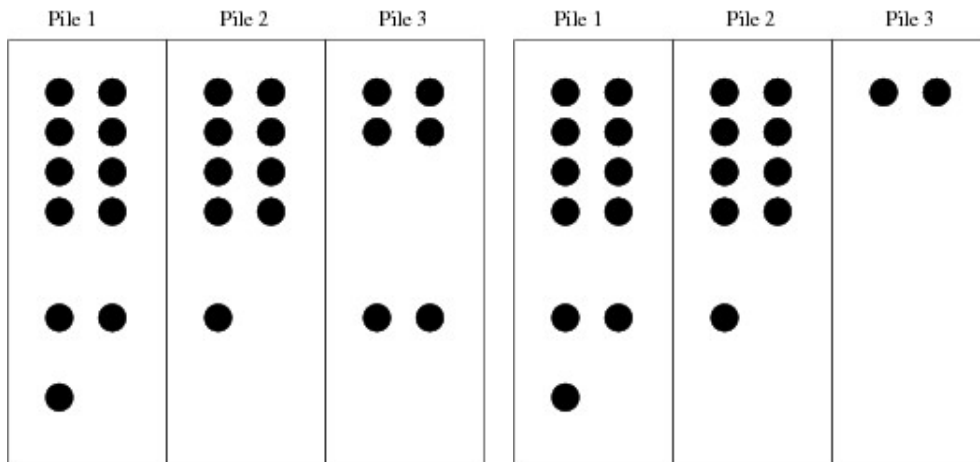
28. Play the game a few times with a partner. Start with small numbers of stones.
29. At what point of the game do you know if you've lost or won? Explain why.
30. Can you find a situation so that you're sure that you will lose? Explain why.
31. Can you find a situation with higher numbers of stones so that you're sure that you will lose? Explain why.
32. Suppose there are 1, 2, and 2 stones in each pile. Do you want to go first or second? Explain why.
33. Suppose there are 1, 2, 3 stones respectively in each pile. Do you want to go first or second? Draw a game tree to explain your strategy.
34. Suppose there are 1, 2, 4 stones respectively in each pile. Do you want to go first or second? Explain why. Draw a game tree to explain your strategy.

**Analyzing a game:** The diagrams below show details of a Three Piles game between Jim and Amanda. The piles hold 11, 9, and 6 stones, respectively. It's Amanda's turn. Notice how she arranges the stones in Figure 5.2(a).

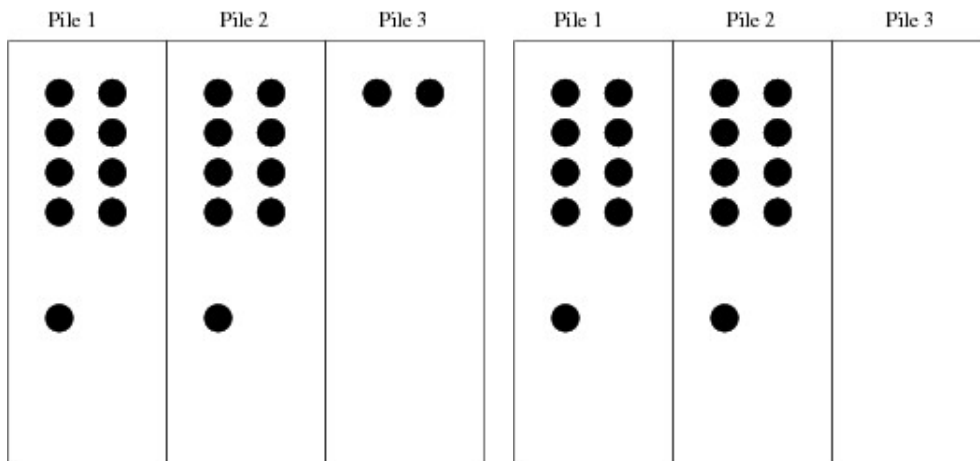
What is Amanda doing? Describe any patterns you notice in the arrangements.

Who is going to win the game? Explain why.

When asked, Amanda reports: "I pair up subpiles." Explain what she might mean by that.



(a) Amanda arranges the three piles. (b) Amanda takes 4 stones from pile three.



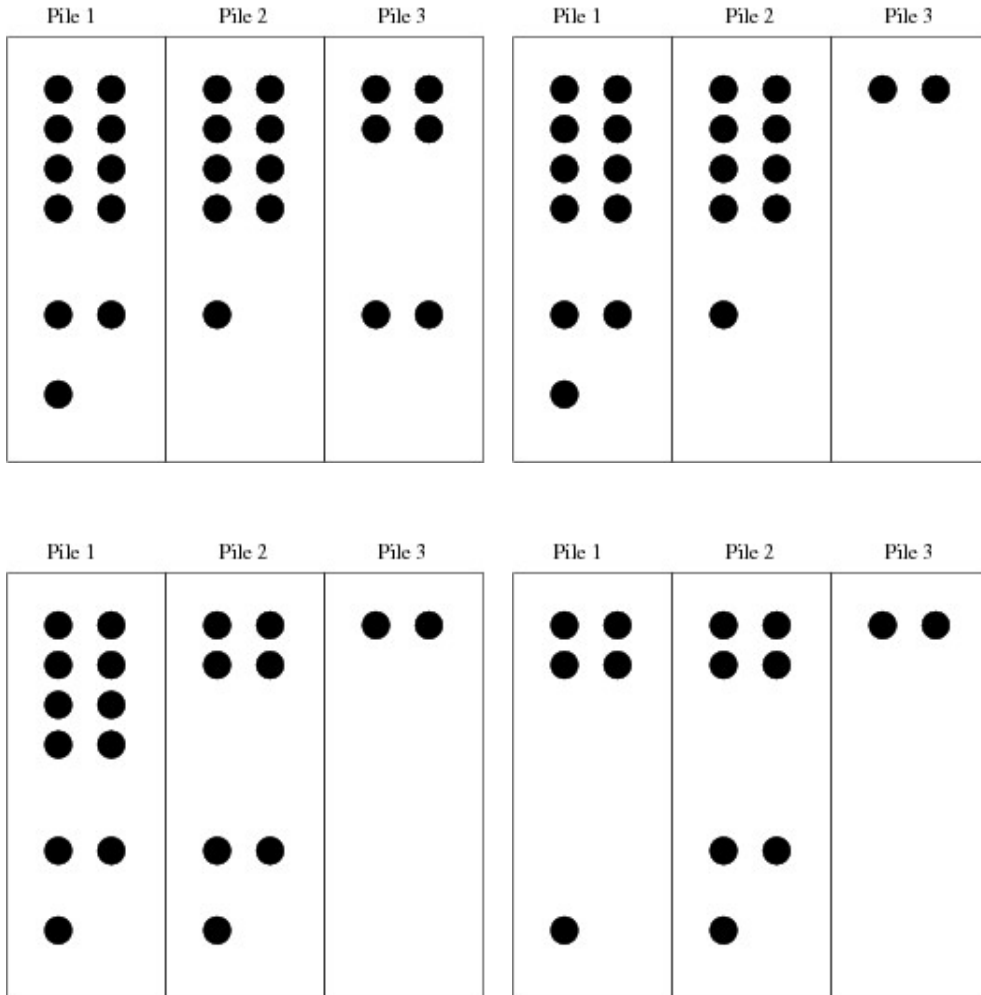
(c) Jim decides to take 2 stones from pile one.

(d) Amanda decides to take two stones from pile three.

Figure 5.2: First game starting with 11-9-6.

In Figures 5.3(a)-5.3(d), we see another game. Amanda starts out in the same way, but now Jim is making different choices.

What is Amanda doing? Describe any patterns you notice in the arrangements.  
 Who is going to win the game? Explain why.  
 What can you say about Amanda's strategy of "pairing up subpiles?" Explain what she might mean by that.



(c) Jim decides to take 2 stones from pile two. (d) Amanda decides to take 6 stones from pile one. Amanda rearranges the piles as shown.

Figure 5.3: Second game of Three Pile between Jim and Amanda, first steps.

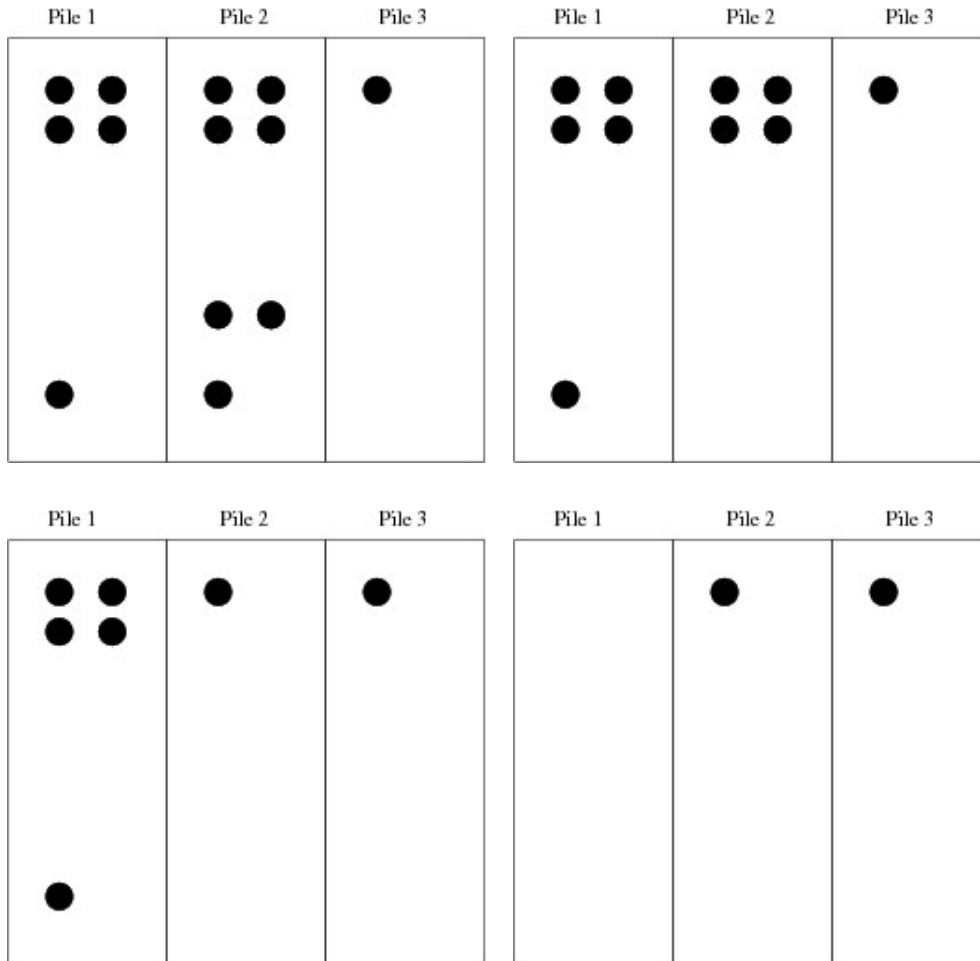


Figure 5.4: Second game of Three Pile between Jim and Amanda, continued.

With your partner, explore some of the strategies you observed. Play a few games, keep track in your notebook of your moves, stone arrangements, and strategies.

Once you feel clear about your strategies, determine your first move for the following arrangements. Explain your strategy.

Three piles with 10, 7, 5 stones, resp.

Three piles with 8, 7, 4 stones, resp.

So far we have seen subpiles of size 1, 2, 4, and 8. How do you think this sequence will continue? Explain.

Carlos does not want to draw pictures into his notebook for a game with 7, 5, 2 stones. Instead, he writes  $7 = 2 + 2 + 2 + 1$ ,  $5 = 4 + 1$ ,  $2 = 2$  to show his subpile arrangement. It is his turn. He is convinced that he will win by taking 4 from the second pile. What do you think is his reasoning? Do you think he is correct? Explain.

His opponent, Amanda writes the remaining subpiles as  $7 = 4 + 2 + 1$ ,  $1 = 1$ ,  $2 = 2$ . She is convinced that *she* will win, by taking 4 from the first pile. Do you think she is correct? Explain.

What important aspect of the winning strategy does this example reveal?

Consider the following game position, with piles of size 15, 25, and 12. Use the table to determine the subpiles, and decide whether you want to go first or second with this game.

Pile size	16	8	4	2	1
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15	0	1	1	1	1
25					
12					

Now consider the following game position, with piles of size 42, 17, and 57. Use the table to determine the subpiles, and decide whether you want to go first or second with this game.

Pile size	16	8	4	2	1
42					
17					
57					

**Writing Assignment:** The family of games we have described so far is actually called *Nim*. Do some research into this game, its history, and its strategies. Write about your findings.

### 5.3.1 Why does it work?

This brief section summarizes the main ingredients that can allow us to prove why the "subpile" strategy will always work.

Every number can be expressed in a unique way as a binary number.

Express piles sizes as binary numbers.

Add each of the slots using *Nim-sum*:  $1 + 1 = 0!$

For each of the games you have worked on so far, determine the Nim-sum at each turn. What kind of patterns do you observe?

**Classroom Discussion:** What is the connection between the subpile strategy and the Nim-sum?

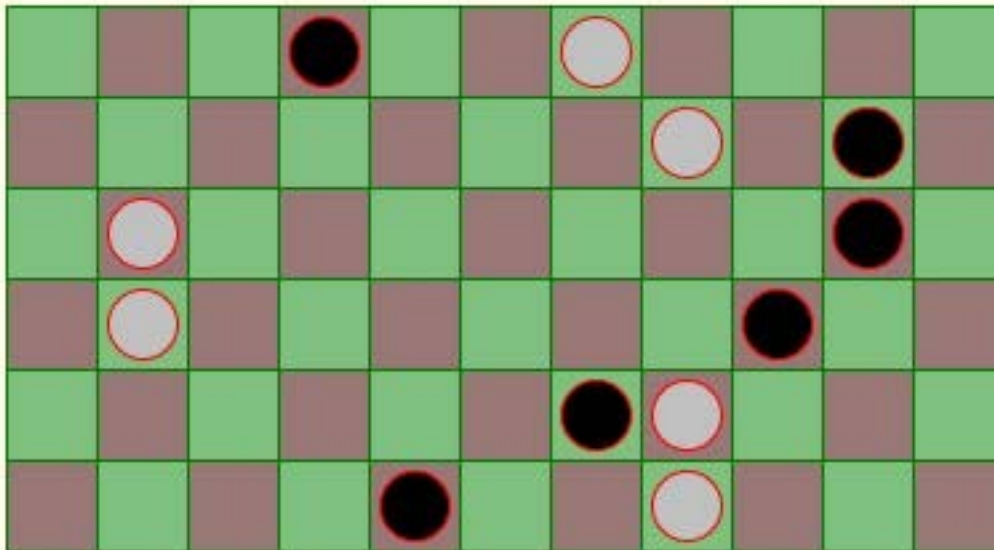


Figure 5.5: Northcott's game position.

**F1.** Northcott's Game: On every row of a rectangular board, there are two checkers: one white and one black. A move consists sliding a single checker in its original row without jumping over another checker. The player to make the last move wins. See Figure 5.5 for an example game board. Use your knowledge of Nim to find a winning strategy for Northcott's game. <sup>1</sup>

<sup>1</sup> For your exploration, you can find online applets of Northcott's game where you can play against a computer.

