# 2.12 The Derivative as a Function

*Multiple Representations*

Given a function $y = f(x)$ , you now know that if you are interested in the instantaneous rate of change of the function at $x = a$, or equivalently the slope of the tangent line to $y = f(x)$ at $x = a$, you can compute the value $f'(a)$ . In all of our examples to date, you have arbitrarily identified a particular value of $a$ as your point of interest: $a = 1$, $a = 3$, etc. But it is not hard to imagine that you may be interested in the derivative value for more than just one $a$-value, and possibly for many of them. In this lesson, you will explore how you can move from computing simply $f'(1)$ or $f'(3)$ to working more generally with $f'(a)$ , and indeed $f'(x)$ . Said differently, you will work toward understanding how the so-called process of "taking the derivative" generates a new function that is derived from the original function $y = f(x)$.

**Investigation 1**: Consider the function $f(x) = 4x - x^2$.

1a) Use the limit definition to compute the following derivative values: $f'(0)$, $f'(1)$, $f'(2)$, and $f'(3)$.

b) Observe that the work to find $f'(a)$ is the same, regardless of the value of a. Based on your work in (a), what do you conjecture is the value of $f'(4)$? How about $f'(5)$ ? (Note: you should not use the limit definition of the derivative to find either value.)

c) Conjecture a formula for $f'(a)$ that depends only on the value $a$. That is, in the same way that you have a formula for $f(x)$ (recall $f(x) = 4x - x^2$), see if you can use your work above to guess a formula for $f'(a)$ in terms of $a$.

## II. How the derivative is itself a function

In your work in Investigation 1 with $f(x) = 4x - x^2$, you may have found several patterns. One comes from observing that $f'(0) = 4$, $f'(1) = 2$, $f'(2) = 0$, and $f'(3) = 2$. That sequence of values leads us naturally to conjecture that $f'(4) = 4$ and $f'(5) = 6$. Even more than these individual numbers, if you consider the role of 0, 1, 2, and 3 in the process of computing the value of the derivative through the limit definition, you observe that the particular number has very little effect on our work. To see this more clearly, let's compute $f'(a)$, where a represents a number to be named later. Following the now standard process of using the limit definition of the derivative,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{4(a+h) - (a+h)^2 - (4a - a^2)}{h}$$

$$= \lim_{h\to 0} \frac{4a + 4h - a^2 - 2ha - h^2 - 4a + a^2}{h}$$

$$= \lim_{h\to 0} \frac{h(4 - 2a - h)}{h}$$

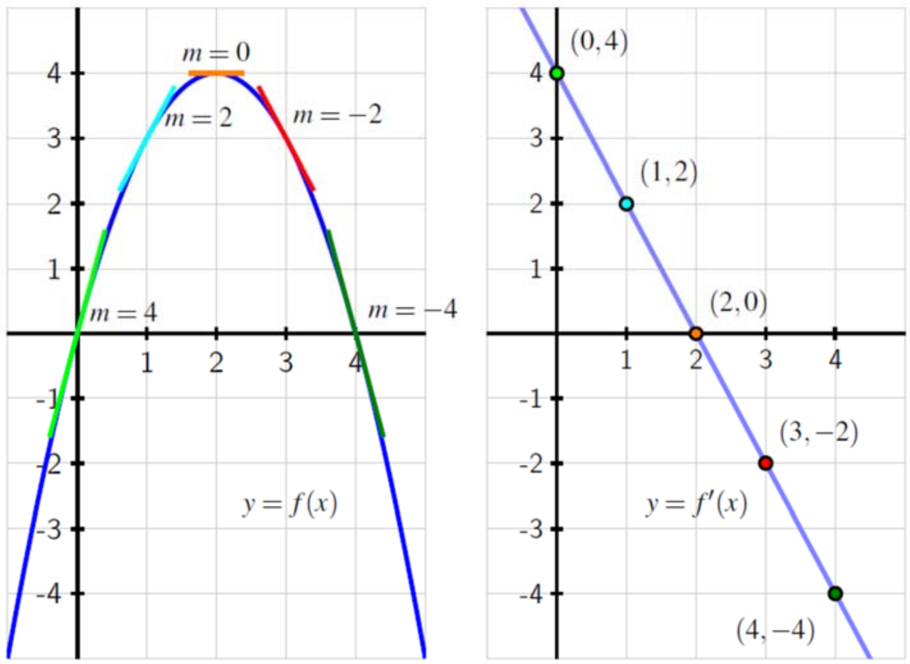
$$= \lim_{h\to 0} \ (4 - 2a - h)$$

Here, observe that neither 4 nor 2a depend on the value of h, so as $h \to 0$, $(4 - 2a - h) \to (4 - 2a)$. Thus, $f'(a) = 4 - 2a$.

This observation is consistent with the specific values you found above: e.g., $f'(3) = 4 - 2(3) = -2$. And indeed, your work with a confirms that while the value of $a$ at which you evaluate the derivative affects the value of the derivative, that value has almost no bearing on the process of computing the derivative. Note further that the letter being used is immaterial: whether you call it $a$, $x$, or anything else, the derivative at a given value is simply given by "4 minus 2 times the value." You choose to use $x$ for consistency with the original function given by $y = f(x)$, as well as for the purpose of graphing the derivative function, and thus you have found that for the function $f(x) = 4x - x^2$, it follows that $f'(x) = 4 - 2x$.

Because the value of the derivative function is so closely linked to the graphical behavior of the original function, it makes sense to look at both of these functions plotted on the same domain.

On the left figure below, is a plot of $f(x) = 4x - x^2$ together with a selection of tangent lines at the points you've considered previously. On the right, is shown a plot of $f'(x) = 4 - 2x$ with emphasis on the heights of the derivative graph at the same selection of points. Notice the connection between colors in the left and right graph: the green tangent line on the original graph is tied to the green point on the right graph in the following way: the slope of the tangent line at a point on the left-hand graph is the same as the height at the corresponding point on the right-hand graph.




The graphs of $f(x) = 4x - x^2$ (at left) and $f'(x) = 4 - 2x$ (at right).
Slopes on the graph of $f$ correspond to heights on the graph of $f'$.

That is, at each respective value of $x$, the slope of the tangent line to the original function at that $x$-value is the same as the height of the derivative function at that $x$-value. Do note, however, that the units on the vertical axes are different: in the left graph, the vertical units are simply the output units of f . On the right-hand graph of $y = f'(x)$, the units on the vertical axis are units of $f$ per unit of $x$.

Of course, this relationship between the graph of a function $y = f(x)$ and its derivative is a dynamic one. Explore the GeoGebra app at [http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_elementary_functions.html](http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_elementary_functions.html) .

In the last lesson, you learned the definition of derivative in terms of a value a to find $f'(a)$. As you have seen, the letter a is merely a placeholder, and it often makes more sense to use $x$ instead. For the record, let's restate the definition of the derivative.

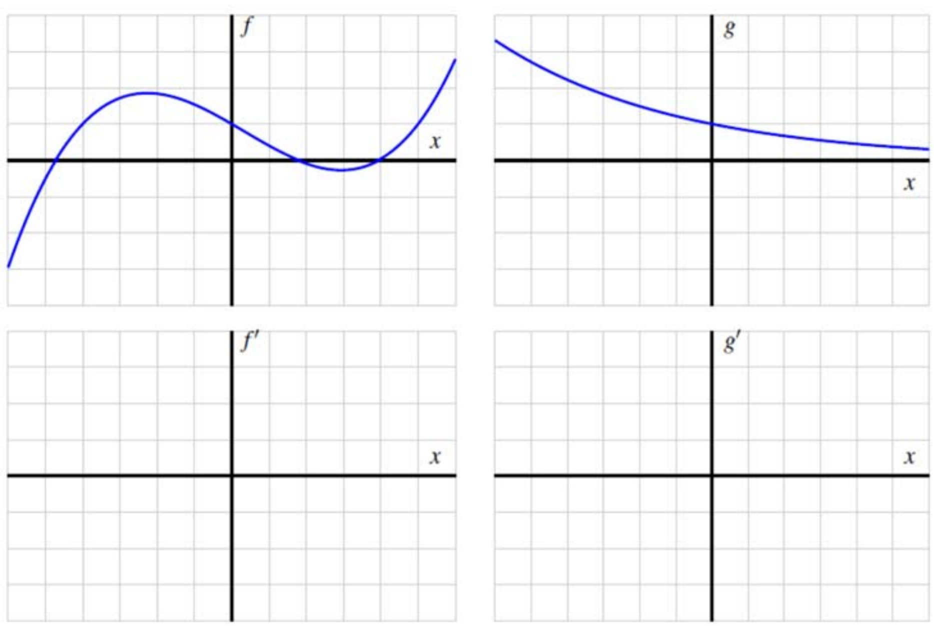
**Definition 2.1A.** Let $f$ be a function and $x = a$, a value in the function's domain. We define the derivative of $f$ with respect to $x$ evaluated at $x = a$, denoted $f'(a)$, by the formula,

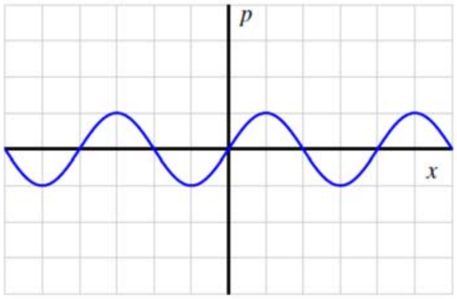
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

You now may take two different perspectives on thinking about the derivative function: given a graph of $y = f(x)$, how does this graph lead to the graph of the derivative function $y = f'(x)$? and given a formula for $y = f(x)$, how does the limit definition of the derivative generate a formula for $y = f'(x)$?

**Investigation 2**: For each given graph of $y = f(x)$, sketch an approximate graph of its derivative function, $y = f'(x)$, on the axes immediately below. The scale of the grid for the graph of f is 1:1; assume the horizontal scale of the grid for the graph of $f'$ is identical to that for $f$. If necessary, adjust and label the vertical scale on the axes for $f'$.





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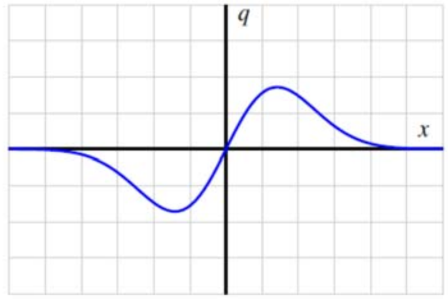

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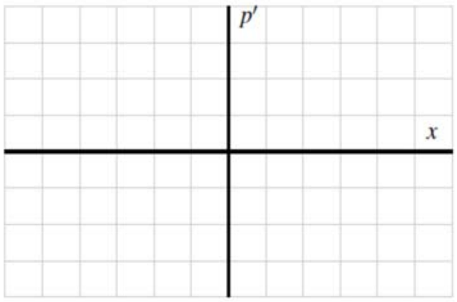

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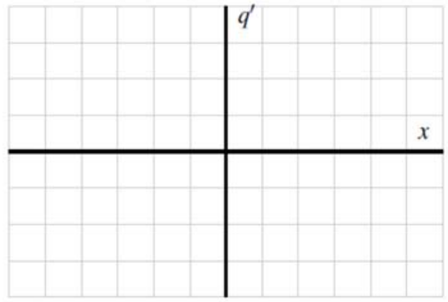

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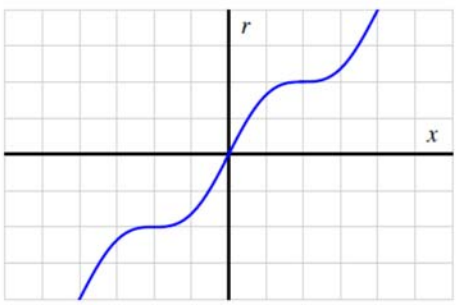

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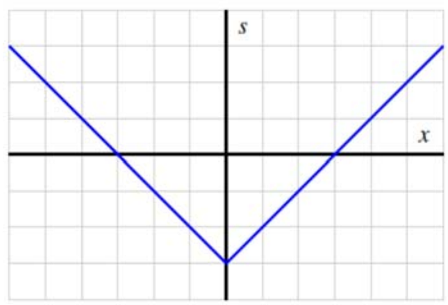

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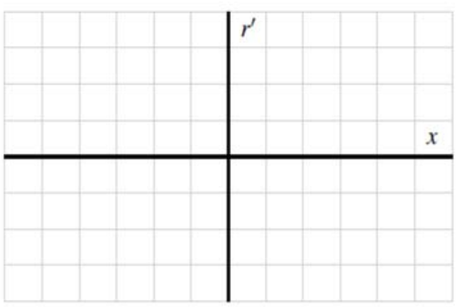

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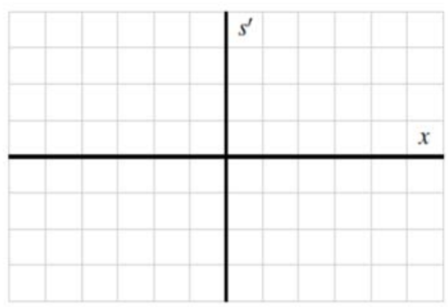

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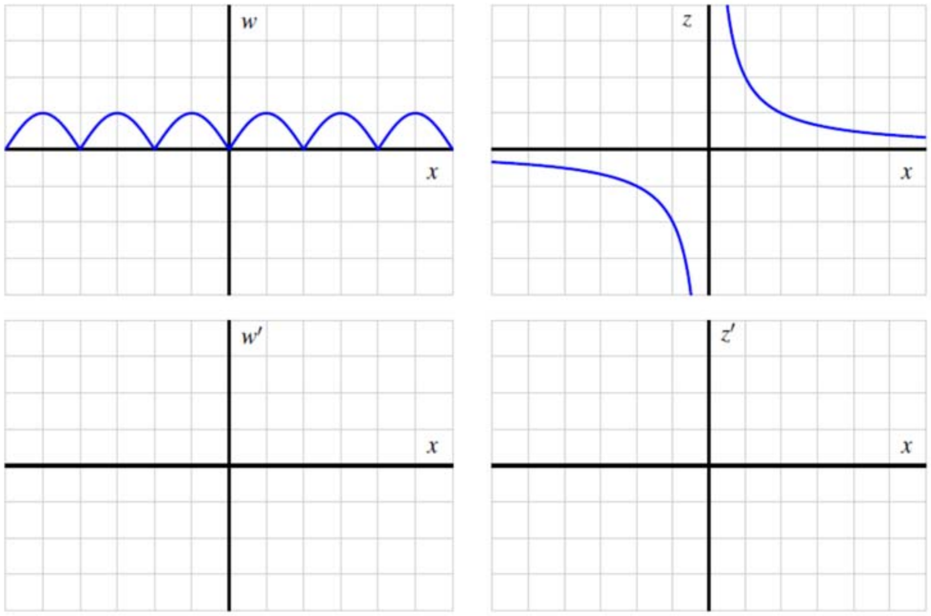



When you are finished with all 8 graphs, write several sentences that describe your overall process for sketching the graph of the derivative function, given the graph the original function. What are the values of the derivative function that you tend to identify first? What do you do thereafter? How do key traits of the graph of the derivative function exemplify properties of the graph of the original function?

For another dynamic challenge try:
http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_try_to_graph.html

Now, recall the opening example of this lesson: you began with the function $f(x) = 4x - x^2$ and used the limit definition of the derivative to show that $f'(a) = 4 - 2a$, or equivalently that $f'(x) = 4 - 2x$. You subsequently graphed the functions $f$ and $f'$. Following Investigation 2, you now understand that you could have constructed a fairly accurate graph of $f'(x)$ without knowing a formula for either $f$ or $f'$. At the same time, it is ideal to know a formula for the derivative function whenever it is possible to find one.

In the next activity, you will explore further the algebraic approach to finding $f'(x)$: given a formula for $y = f(x)$, the limit definition of the derivative will be used to develop a formula for $f'(x)$.

**Investigation 3**: For each of the listed functions, determine a formula for the derivative function. For the first two, determine the formula for the derivative by thinking about the nature of the given function and its slope at various points; do not use the limit definition.

For the latter four, use the limit definition. Pay careful attention to the function names and independent variables. It is important to be comfortable with using letters other than $f$ and $x$. For example, given a function $p(z)$, you call its derivative $p'(z)$.

3a) $f(x) = 1$

b) $g(t) = t$

c) $p(z) = z^2$

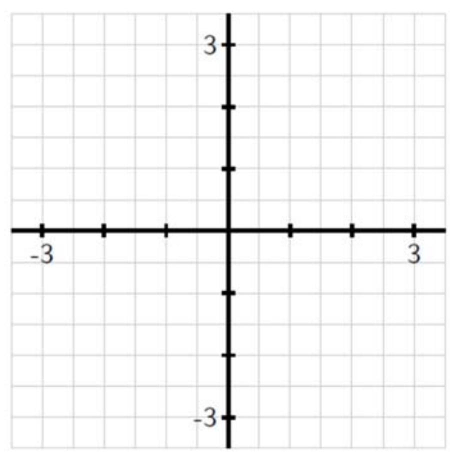
d) $q(s) = s^3$

e) $F(t) = \frac{1}{t}$

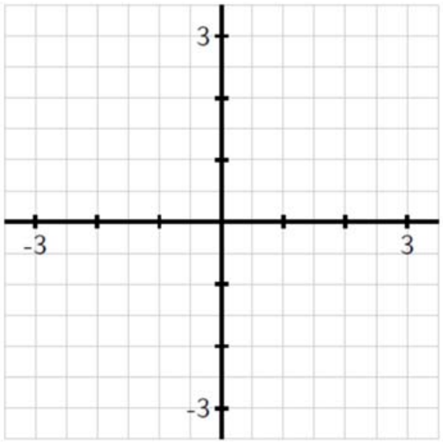
f) $G(y) = \sqrt{y}$

## III. Exercises

1. Let $f$ be a function with the following properties: f is differentiable at every value of $x$ (that is, $f$ has a derivative at every point), $f(-2) = 1$, and
$f'(-2) = -2$, $f'(-1) = -1$, $f'(0) = 0$,
$f'(1) = 1$ and $f'(2) = 2$.




a) On the axes provided at right, sketch a possible graph of $y = f(x)$. Explain why your graph meets the stated criteria.




b) On the axes at left, sketch a possible graph of $y = f\,'(x)$. What type of curve does the provided data suggest for the graph of $y = f\,'(x)$?

c) Conjecture a formula for the function $y = f(x)$ . Use the limit definition of the derivative to determine the corresponding formula for $y = f\,'(x)$ . Discuss both graphical and algebraic evidence for whether or not your conjecture is correct.

2. Consider the function $g(x) = x^2 - x + 3$.

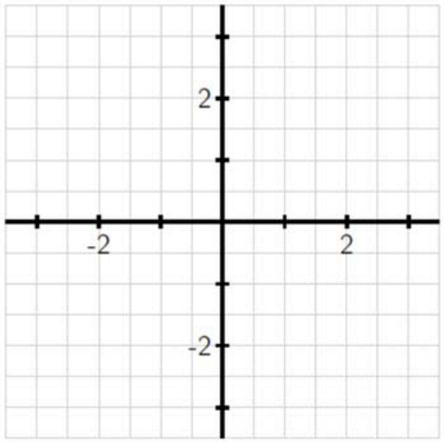
a) Use the limit definition of the derivative to determine a formula for $g'(x)$.

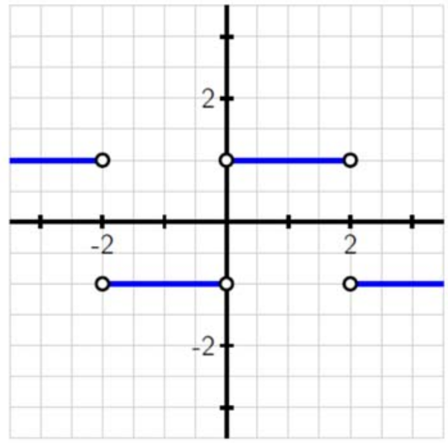
b) Use a graphing utility to plot both $y = g(x)$ and your result for $y = g'(x)$; does your formula for $g'(x)$ generate the graph you expected?

c) Use the limit definition of the derivative to find a formula for $p'(x)$ where $p(x) = 5x^2 - 4x + 12$.

d) Compare and contrast the formulas for g'(x) and p'( )x you found. How do the constants 5, 4, 12, and 3 affect the results?

3. Let $g$ be a continuous function (that is, one with no jumps or holes in the graph) and suppose that a graph of $y = g'(x)$ is given by the graph below right.





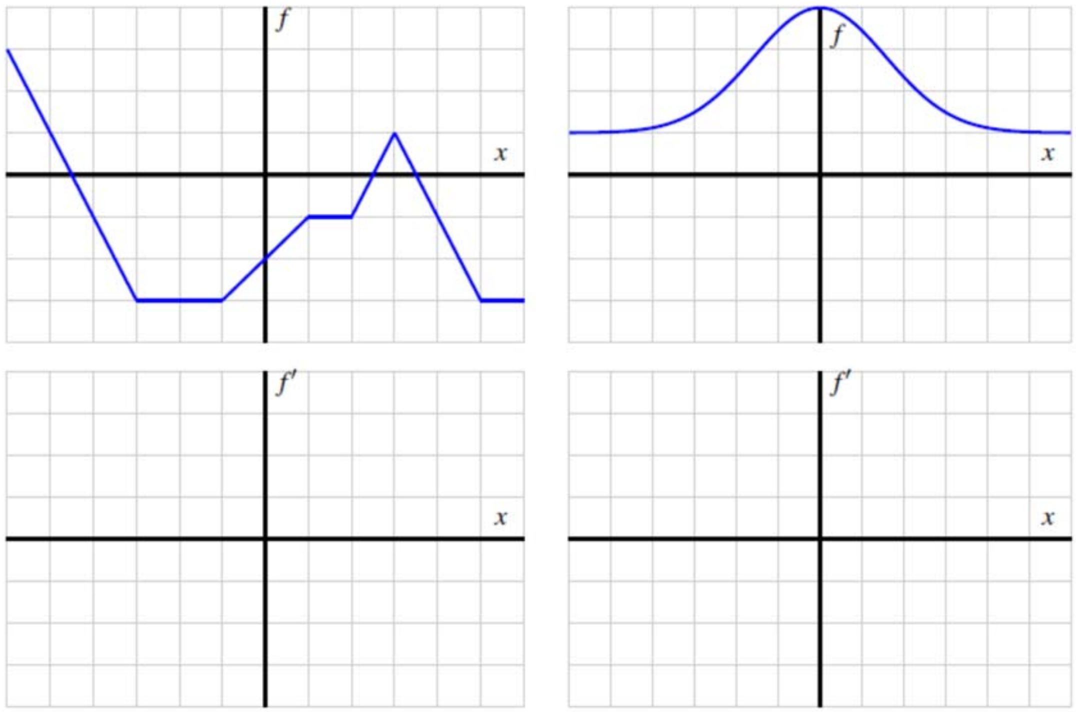

a) Observe that for every value of $x$ that satisfies $0 < x < 2$, the value of $g'(x)$ is constant. What does this tell you about the behavior of the graph of $y = g\ (x)$ on this interval?

b) On what intervals other than $0 < x < 2$ do you expect $y = g\ (x)$ to be a linear function? Why?

c) At which values of $x$ is $g'(x)$ not defined? What behavior does this lead you to expect to see in the graph of $y = g(x)$?

d) Suppose that $g(0) = 1$. On the axes provided above left, sketch an accurate graph of $y = g(x)$.

4. For each graph that provides an original function $y = f\,(x)$ in the figure on the following page, your task is to sketch an approximate graph of its derivative function, $y = f\,'(x)$ , on the axes immediately below. View the scale of the grid for the graph of $f$ as being 1:1, and assume the horizontal scale of the grid for the graph of $f\,'$ is identical to that for $f$ . If you need to adjust the vertical scale on the axes for the graph of $f\,'$ you should label that accordingly.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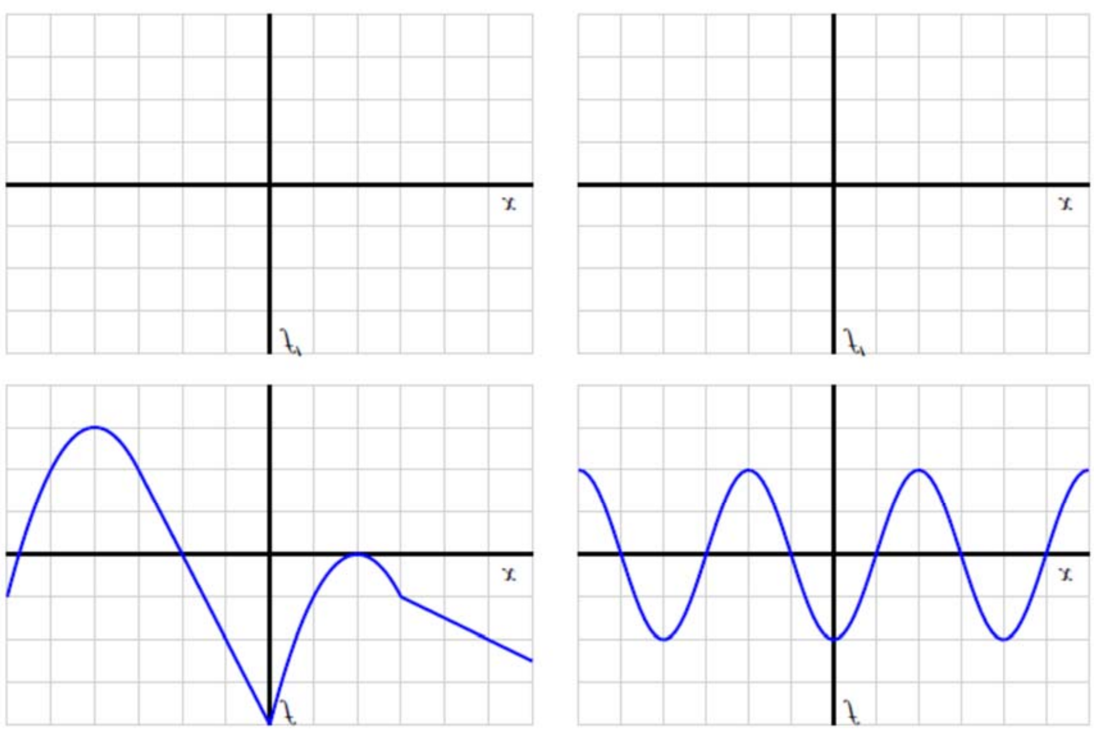

## IV. Assessment – Khan Academy

1. Complete the following practice exercises from Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/derivative-introduction-ab/derivative-as-a-function-ab/e/visualizing_derivatives