2.21 Power Up!

Computing Basic Differentiation

In the last few lessons, you explored the concept of the derivative of a function. You now know that the derivative $f'$of a function f measures the instantaneous rate of change of f with respect to x as well as the slope of the tangent line to $y = f(x)$ at any given value of x. To date, you have focused primarily on interpreting the derivative graphically or, in the context of functions in a physical setting, as a meaningful rate of change. To actually calculate the value of the derivative at a specific point, you have typically relied on the limit definition of the derivative.

In this present chapter, you will investigate how the limit definition of the derivative,

$$F'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

leads to interesting patterns and rules that enable you to quickly find a formula for $f'(x)$ based on the formula for $f(x)$ without using the limit definition directly. For example, you already know that if $f(x) = x$, then it follows that $f'(x) = 1$. While you could use the limit definition of the derivative to confirm this, you know it to be true because $f(x)$ is a linear function with slope 1 at every value of x. One of your goals is to be able to take standard functions, say ones such as $g(x) = 4x^7 - \sin x + 3e^x$, and, based on the algebraic form of the function, be able to apply shortcuts to almost immediately determine the formula for $g'(x)$.

Investigation1: Functions of the form $f(x) = x^n$, where $n = 1, 2, 3, \ldots$, are often called power functions. The first two questions below revisit work you did earlier in Chapter 1, and the following questions extend those ideas to higher powers of x.

1a) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^2$.

b) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^3$.

c) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^4$. (Hint: $(a + b)^4 = a^4 + 4a^3b + 6a^2 b^2 + 4ab^3 + b^4$. Apply this rule to expand $(x + h)^4$ within the limit definition.)

d) Based on your work in (a), (b), and (c), what do you conjecture is the derivative of $f(x) = x^5$? Of $f(x) = x^{13}$?

e) Conjecture a formula for the derivative of $f(x) = x^n$ that holds for any positive integer n. That is, given $f(x) = x^n$ where n is a positive integer, what do you think is the formula for $f'(x)$?

II. Some Key Notation

In addition to your usual f' notation for the derivative, there are other ways to symbolically denote the derivative of a function, as well as the instruction to take the derivative. You know that if you have a function, say $f(x) = x^2$, that you can denote its derivative by $f'(x)$, and you write $f'(x) = 2x$. Equivalently, if you are thinking more about the relationship between y and x, you sometimes denote the derivative of y with respect to x with the symbol

$$\frac{dy}{dx}$$

which you read "dee-y dee-x." This notation comes from the fact that the derivative is related to the slope of a line, and slope is measured by $\frac{\Delta y}{\Delta x}$. Note that while you read $\frac{\Delta y}{\Delta x}$ as "change in y over change in x," for the derivative symbol $\frac{dy}{dx}$, you view this is a single symbol, not a quotient of two quantities[1]. For example, if $y = x^2$, you'll write that the derivative is $\frac{dy}{dx} = 2x$.

Furthermore, you will use a variant of $\frac{dy}{dx}$ notation to convey the instruction to take the derivative of a certain quantity with respect to a given variable. In particular, if you write

$$\frac{d}{dx}[\blacksquare]$$

this means "take the derivative of the quantity in $\blacksquare$ with respect to x." To continue the example above with the squaring function, here you may write

$$\frac{d}{dx}[x^2] = 2x.$$

[1] That is, one does **not** say "dee-y over dee-x."

It is important to note that the independent variable can be different from x. If you have $f(z) = z^2$, you then write $f'(z) = 2z$. Similarly, if $y = t^2$, you may write $\frac{dy}{dx} = 2t$ or $\frac{d}{dt}[t^2] = 2t$.

This notation may also be applied to second derivatives: $y'' = \frac{d^2y}{dx^2}$.

III. Constant, Power, and Exponential Functions

So far, you know the derivative formula for two important classes of functions: constant functions and power functions. For the first kind, observe that if $f(x) = c$ is a constant function, then its graph is a horizontal line with slope zero at every point. Thus, $\frac{d}{dx}[c] = 0$. .

We summarize this with the following rule.

Constant Functions: For any real number c, if $f(x) = c$, then $f'(x) = 0$.

Thus, if $f(x) = 7$, then $f'(x) = 0$. Similarly, $\frac{d}{dx}\left[\sqrt{3}\right] = 0$.

For power functions, from your work in Investigation 1, you have conjectured that for any positive integer n, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. Not only can this rule be formally proved to hold for any positive integer n, but also for any nonzero real number (positive or negative).

Power Functions: For any nonzero real number, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

This rule for power functions allows you to find derivatives such as the following: if $g(z) = z^{-3}$, then $g'z) = -3z^{-4}$. Similarly, if $h(t) = t^{7/5}$, then $\frac{dy}{dx} = \frac{7}{5}t^{2/5}$.

As you next turn to thinking about derivatives of combinations of basic functions, it will be instructive to have one more type of basic function whose derivative formula you know.

Exponential Functions: For any positive real number a, if $f(x) = a^x$, then $f'(x) = a^x \ln(a)$.

For instance, this rule says if $f(x) = 2x$, then $f'(x) = 2x \ln(2)$. Similarly, for $p(t) = 10^t$, $p'(t) = 10^2 \ln(10)$.

It is especially important to note that when $a = e$, where e is the base of the natural logarithm function, you get

$$\frac{d}{dx}[e^x] = e^x \ln(e) = e^x$$

since $\ln(e) = 1$. This is an extremely important property of the function e^x: its derivative function is itself!

Finally, note carefully the distinction between power functions and exponential functions: in power functions, the variable is in the base, as in x^2, while in exponential functions, the variable is in the power, as in 2^x. As you can see from the rules, this makes a big difference in the form of the derivative.

Investigation2: Use the three rules above to determine the derivative of each of the following functions. For each, state your answer using full and proper notation, labeling the derivative with its name. For example, if you are given a function $h(z)$, you should write "h'(z)=" or "$\frac{dy}{dx} =$" as part of your answer.

2a) $f(t) = \pi$

b) $g(z) = 7^z$

c) $h(w) = w^{3/4}$

d) $p(x) = 3^{1/2}$

e) $r(t) = (\sqrt{2})^t$

f) $\frac{d}{dq}[q^{-1}]$

(g) $m(t) = \frac{1}{t^3}$

IV. Constant Multiples and Sums of Functions

Of course, most of the functions you encounter in mathematics are more complicated than being simply constant, a power of a variable, or a base raised to a variable power. In this section and several following, you will learn how to quickly compute the derivative of a function constructed as an algebraic combination of basic functions. For instance, you'd like to be able to understand how to take the derivative of a polynomial function such as $p(t) = 3t^5 - 7t^4 + t^2 - 9$, which is a function made up of constant multiples and sums of powers of t. To that end, examine two new rules: the Constant Multiple Rule and the Sum Rule.

Say you have a function $y = f(x)$ whose derivative formula is known. How is the derivative of $y = k\,f(x)$ related to the derivative of the original function? Recall that when you multiply a function by a constant k, you vertically stretch the graph by a factor of k (and reflect the graph across $y = 0$ if $k < 0$). This vertical stretch affects the slope of the graph, making the slope of the function $y = k\,f(x)$ be k times as steep as the slope of $y = f(x)$. In terms of the derivative, this is essentially saying that when you multiply a function by a factor of k, you change the value of its derivative by a factor of k as well. Thus, the Constant Multiple Rule holds:

> **Constant Multiple Rule:** For any real number k, if $f(x)$ is a differentiable function with derivative $f'(x)$, then $\frac{d}{dx}[kf(x)] = kf'(x)$.

In words, this rule says that "the derivative of a constant times a function is the constant times the derivative of the function." For example, if $g(t) = 3 \cdot 5t$, you have $G'(t) = 3 \cdot 5t \ln(5)$.

Next, examine what happens when you take a sum of two functions.

> **The Sum Rule:** If $f(x)$ and $g(x)$ are differentiable functions with derivatives $f'(x)$ and $f'(x)$ respectively, then $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$.

In words, the Sum Rule tells us that "the derivative of a sum is the sum of the derivatives." It also tells us that any time you take a sum of two differentiable functions, the result must also be differentiable. Furthermore, because you can view the difference function $y = (f - g)(x) = f(x) - g(x)$ as $y = f(x) + (-1 \cdot g(x))$, the Sum

Rule and Constant Multiple Rules together tell us that

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x),$$

or that "the derivative of a difference is the difference of the derivatives."

Investigation 3: Use only the rules for constant, power, and exponential functions, together with the Constant Multiple and Sum Rules, to compute the derivative of each function below with respect to the given independent variable. Note well that you do not yet know any rules for how to differentiate the product or quotient of functions. This means that you may have to do some algebra first on the functions below before you can use existing rules to compute the desired derivative formula. In each case, label the derivative you calculate with its name using proper derivative notation.

a) $f(x) = x^{5/3} - x^4 + 2^x$

b) $g(x) = 14e^x + 3x^5 - x$

c) $h(z) = \sqrt{z} + \frac{1}{z^4} + 5^z$

d) $r(t) = \sqrt{53}t^7 - \pi e^t + e^4$

e) $s(y) = (y^2 + 1)(y^2 - 1)$

(f) $q(x) = \frac{x^3 - x + 2}{x}$

(g) $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$

In the same way that you have shortcut rules to help us find derivatives, you introduce some language that is simpler and shorter. Often, rather than say "take the derivative of *f*," you'll instead say simply "differentiate *f*." This phrasing is tied to the notion of having a derivative to begin with: if the derivative exists at a point, you say "*f* is differentiable," which is tied to the fact that *f* can be differentiated.

As you work more and more with the algebraic structure of functions, it is important to strive to develop a big picture view of what you are doing. Here, you can note several general observations based on the rules you have so far. One is that the derivative of any polynomial function will be another polynomial function, and that the degree of the derivative is one less than the degree of the original function. For instance, if $p(t) = 7t^5\ 4t^3 + 8t$, p is a degree 5 polynomial, and its derivative, $p'(t) = 35t^4 - 12t^2 + 8$, is a degree 4 polynomial.

Additionally, the derivative of any exponential function is another exponential function: for example, if $g(z) = 7 \cdot 2^z$, then $g'(z) = 7 \cdot 2^z \ln(2)$, which is also exponential.

Investigation 4: Each of the following questions asks you to use derivatives to answer key questions about functions. Be sure to think carefully about each question and to use proper notation in your responses.

a) Find the slope of the tangent line to $h(z) = \sqrt{z} + \frac{1}{z}$ at the point where $z = 4$.

b) A population of cells is growing in such a way that its total number in millions is given by the function $P(t) = 2(1.37)^t + 32$, where t is measured in days.

> i. Determine the instantaneous rate at which the population is growing on day 4, and include units on your answer.
>
> ii. Is the population growing at an increasing rate or growing at a decreasing rate on day 4? Explain.

c) Find an equation for the tangent line to the curve $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$ at the point where $a = -1$.

d) What is the difference between being asked to find the slope of the tangent line (asked in (a)) and the equation of the tangent line (asked in (c))?

V. Exercises

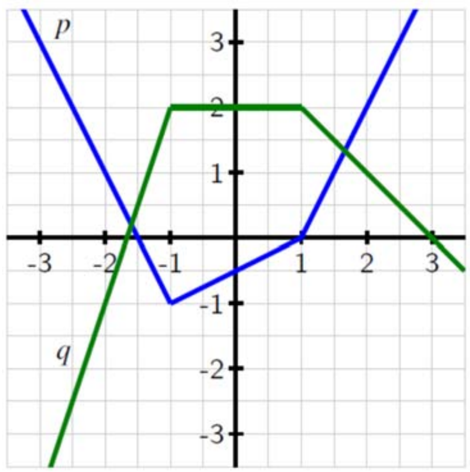
1. Let f and g be differentiable functions for which the following information is known: $f(2) = 5$, $g(2) = -3$, $f'(2) = -1/2$, $g'(2) = 2$.

> a) Let h be the new function defined by the rule $h(x) = 3f(x) - 4g(x)$. Determine $h(2)$ and $h'(2)$.
>
> b) Find an equation for the tangent line to $y = h(x)$ at the point $(2, h(2))$.
>
> c) Let p be the function defined by the rule $p(x) = -2f(x) + \frac{1}{2}g(x)$. Is p

increasing, decreasing, or neither at $a = 2$? Why?

d) Estimate the value of $p(2.03)$ by using the local linearization of p at the point $(2, p(2))$.

2. Let functions p and q be the piecewise linear functions given by their respective graphs below. Use the graphs to answer the following questions.



a) At what values of x is p not differentiable? At what values of x is q not differentiable? Why?

b) Let $r(x) = p(x) + 2q(x)$. At what values of x is r not differentiable? Why?

c) Determine r '(-2) and r '(0).

d) Find an equation for the tangent line to $y = r(x)$ at the point $(2, r(2))$.

3. Consider the functions $r(t) = t^t$ and $s(t) = \arccos(t)$, for which you are given the facts that $r'(t) = t^t(\ln(t) + 1)$ and $s'(t) = -\frac{1}{\sqrt{1-t^2}}$. Do not be concerned with where these derivative formulas come from. You restrict your interest in both functions to the domain $0 < t < 1$.

a) Let $w(t) = 3t^t - 2\arccos(t)$. Determine $w'(t)$.

b) Find an equation for the tangent line to $y = w(t)$ at the point $(1, w(1))$.

c) Let $v(t) = t^t + \arccos(t)$. Is v increasing or decreasing at the instant $t = 1$? Why?

4. Let $f(x) = a^x$. The goal of this problem is to explore how the value of a affects the derivative of $f(x)$, without assuming you know the rule for $\frac{d}{dx}[a^x]$ that you have stated and used in earlier work in this section.

a) Use the limit definition of the derivative to show that

$$f'(x) = \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

(b) Explain why it is also true that

$$f'(x) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$

(c) Use computing technology and small values of h to estimate the value of

$$L = \lim_{h \to 0} \frac{a^h - 1}{h}$$

when $a = 2$. Do likewise when $a = 3$.

d) Note that it would be ideal if the value of the limit L was 1, for then f would be a particularly special function: its derivative would be simply a^x, which would mean that its derivative is itself. By experimenting with different values of a between 2 and 3, try to find a value for a for which

$$L = \lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

e) Compute $\ln(2)$ and $\ln(3)$. What does your work in (b) and (c) suggest is true about $\frac{d}{dx}[2^x]$ and $\frac{d}{dx}[3^x]$?

f) How do your investigations in (d) lead to a particularly important fact about the function $f(x) = e^x$?

VI. Practice – Khan Academy

1. Complete the first seven online practice exercises in the sixth unit (Basic Differentiation) of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/basic-differentiation-ab?t=practice

II. Optional

1. None