# 2.26 Anything You Can Do… I Can Un-Do

## *Derivatives of Inverse Functions*

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One of the most important functions in all of mathematics is the natural exponential function $f(x) = e^x$. Because the natural logarithm, $g(x) = \ln x$ , is the inverse of the natural exponential function, the natural logarithm is similarly important. One of your goals in this section is to learn how to differentiate the logarithm function, and thus expand your library of basic functions with known derivative formulas. First, you investigate a more familiar setting to refresh some of the basic concepts surrounding functions and their inverses.

**Investigation 1**: The equation $y = \frac{5}{9}(x - 32)$ relates a temperature given in $x$ degrees Fahrenheit to the corresponding temperature y measured in degrees Celcius.

1a) Solve the equation $y = \frac{5}{9}(x - 32)$ for $x$ to write $x$ (Fahrenheit temperature) in terms of $y$ (Celcius temperature).

b) Let $C(x) = \frac{5}{9}(x - 32)$ be the function that takes a Fahrenheit temperature as input and produces the Celcius temperature as output. In addition, let $F(y)$ be the function that converts a temperature given in y degrees Celcius to the temperature $F(y)$ measured in degrees Fahrenheit. Use your work in (a) to write a formula for $F(y)$.

c) Next consider the new function defined by $p(x) = F(C(x)$. Use the formulas for F and $C$ to determine an expression for $p(x)$ and simplify this expression as much as possible. What do you observe?

d) Now, let $r(y) = C(F(y))$. Use the formulas for $F$ and $C$ to determine an expression for $r(y)$ and simplify this expression as much as possible. What do you observe?

(e) What is the value of $C'(x)$? of $F'(y)$? How do these values appear to be related?

## II. Basic facts about inverse functions

A function $f: A \rightarrow B$ is a rule that associates each element in the set $A$ to one and only one element in the set $B$. You call $A$ the domain of $f$ and $B$ the codomain of $f$. If there exists a function $g: B \rightarrow A$ such that $g\,(f(\,a))\ = $ a for every possible choice of a in the set $A$ and $f\,(g(b)) = b$ for every $b$ in the set $B$, then you say that g is the inverse of $f$. You often use the notation $f^{-1}$ (read " f -inverse") to denote the inverse of $f$. Perhaps the most essential thing to observe about the inverse function is that it undoes the work of $f$.

Indeed, if $y = f\,(x)$, then

$$f^{-1}(y) = f^{-1}(f\,(x)) = x$$

and this leads you to another key observation: writing $y = f\,(x)$ and $x = f^{-1}(y)$ say the exact same thing. The only difference between the two equations is one of perspective – one is solved for x, while the other is solved for $y$.

Here you briefly remind yourselves of some key facts about inverse functions. For a function $f: A \rightarrow B$,

- $f$ has an inverse if and only if f is one-to-one[1] and onto[2];
- provided $f^{-1}$ exists, the domain of $f^{-1}$ is the codomain of $f$, and the codomain of $f^{-1}$ is the domain of $f$;
- $f^{-1}(f\,(x) = x$ for every x in the domain of f and $f^{-1}(f\,(y) = y$ for every $y$ in the codomain of $f$;
- $y = f\,(x)$ if and only if $x = f^{-1}(y)$.

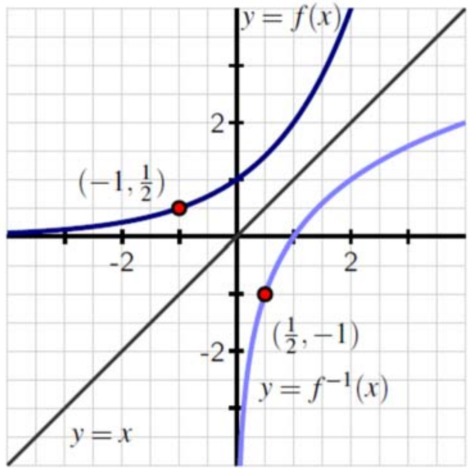
The last stated fact reveals a special relationship between the graphs of $f$ and $f^{-1}$. In particular, if you consider $y = f\,(x)$ and a point $(x, y)$ that lies on the graph of f , then it is also true that $x = f^{-1}(y)$, which means that the point $(y, x)$ lies on the graph of $f^{-1}$.

This shows you that the graphs of $f$ and $f^{-1}$ are the reflections of one another across the line y = x, since reflecting across $y = x$ is precisely the geometric action that swaps the coordinates in an ordered pair. In the figure below, you see this

[1] A function $f$ is *one-to-one* provided that no two distinct inputs lead to the same output.

[2] A function f is *onto* provided that every possible element of the codomain can be realized as an output of the function for some choice of input from the domain.

exemplified for the function $f(x) = 2^x$ and its inverse, with the points $(-1, \frac{1}{2})$ and $(\frac{1}{2}, -1)$ highlighting the reflection of the curves across $y = x$.




To close this review of important facts about inverses, recall that the natural exponential function $f(x) = e^x$ has an inverse function, and its inverse is the natural logarithm, $x = f^{-1}(y) = \ln(y)$. Indeed, writing $y = e^x$ is interchangeable with $x = \ln(y)$, plus $\ln(e^x) = x$ for every real number $x$ and $e^{\ln(y)} = y$ for every positive real number $y$.

## III. The derivative of the natural logarithm function

In what follows, you determine a formula for the derivative of $g(x) = \ln(x)$ . To do so, you take advantage of the fact that you know the derivative of the natural exponential function, which is the inverse of $g$. You know that writing $g(x) = \ln x$ is equivalent to writing $e^{g(x)} = x$. If you differentiate both sides of this most recent equation, observe that

$$\frac{d}{dx}\left[e^{(g(x)}\right] = \frac{d}{dx}[x]$$

The right-hand side is simply 1; applying the chain rule to the left side, you find that

$$e^{(g(x)}g'(x) = 1$$

Solve for $g'(x)$,

$$g'(x) = \frac{1}{e^{(g(x)}}$$

Finally, recall that since $g(x) = \ln(x)$, $e^{(g(x)} = e^{\ln(x)}$, and thus

$$g'(x) = \frac{1}{x}$$

**Natural Logarithm:** For all positive real number $x$, $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

This rule for the natural logarithm function now joins your list of other basic derivative rules that you have already established. There are two particularly interesting things to note about the fact that $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$. One is that this rule is restricted to only apply to positive values of x, as these are the only values for which the original function is defined. The other is that for the first time in your work, differentiating a basic function of a particular type has led to a function of a very different nature: the derivative of the natural logarithm is not another logarithm, nor even an exponential function, but rather a rational one.

Derivatives of logarithms may now be computed in concert with all of the rules known to date. For instance, if $f(t) = \ln(t^2 + 1)$, then by the chain rule, $f'(t) = \frac{1}{t^2+1} \cdot 2t$.

**Investigation 2**: For each function given below, find its derivative.

2a) $h(x) = x^2 \ln(x)$

b) $p(t) = \frac{\ln(t)}{e^t+1}$

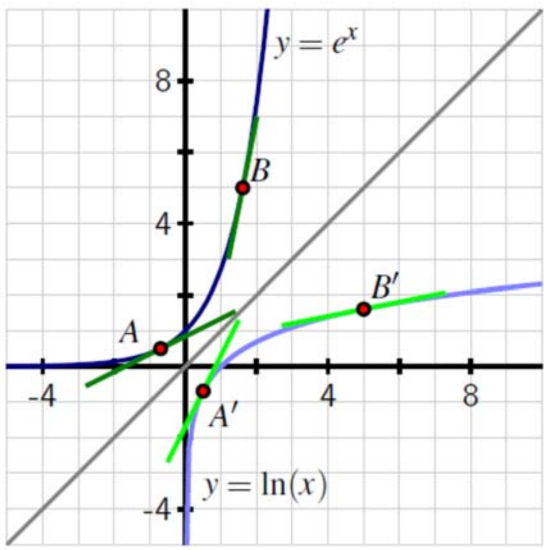
c) $s(y) = \ln(\cos(y) + 2)$

d) $z(x) = \tan(\ln(x))$

e) $m(z) = \ln(\ln(z))$

In addition to the important rule you have derived for the derivative of the natural log functions, there are additional interesting connections to note between the graphs of $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$.

In the figure below, you are reminded that since the natural exponential function has the property that its derivative is itself, the slope of the tangent to $y = e^x$ is equal to the height of the curve at that point. For instance, at the point $A = (\ln(0.5), 0.5)$, the slope of the tangent line is $m_A = 0.5$, and at $B = (\ln(5), 5)$, the tangent line's slope is $m_B = 5$. At the corresponding points *A*, and *B*, on the graph of the natural logarithm function (which come from reflecting across the line $y = x$), you know that the slope of the tangent line is the reciprocal of the *x*-coordinate of the point (since $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$). Thus, with $A' = (0.5, \ln(0.5))$, you have $m_{A'} = \frac{1}{0.5} = 2$, and at $B' = (5, \ln(5))$, $m_{B'} = \frac{1}{5}$.




In particular, you observe that $m_{A'} = \frac{1}{m_A}$ and $m_{B'} = \frac{1}{m_B}$. This is not a coincidence, but in fact holds for any curve $y = f(x)$ and its inverse, provided the inverse exists. One rationale for why this is the case is due to the reflection across $y = x$: in so doing, you essentially change the roles of *x* and *y*, thus reversing the rise and run, which leads to the slope of the inverse function at the reflected point being the reciprocal of the slope of the original function.

## IV. Inverse trigonometric functions and their derivatives

Trigonometric functions are periodic, so they fail to be one-to-one, and thus do not have inverses. However, if we restrict the domain of each trigonometric function, you can force the function to be one-to-one. For instance, consider the sine function on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Because no output of the sine function is repeated on this interval, the function is one-to-one and thus has an inverse. In particular, if you view $f(x) = \sin(x)$ as having domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and codomain [-1, 1], then there exists an inverse function f  such that

$$f^{-1}: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

You call $f^{-1}$ the ***arcsine*** (or inverse sine) function and write $f^{-1} = \arcsin(x)$. It is especially important to remember that writing

$$y = \sin(x) \text{ and } x = \arcsin(y)$$

say the exact same thing. You often read "the arcsine of $y$" as "the angle whose sine is $y$." For example, you say that $\frac{\pi}{6}$ is the angle whose sine is 1 , which can be written more concisely as $\arcsin(\ 1\ ) = \frac{\pi}{6}$, which is equivalent to writing $\sin(\frac{\pi}{6}) = 1$ .

Next, you will determine the derivative of the arcsine function. Letting $h(x) = \arcsin(x)$, your goal is to find $h'(x)$. Since $h(x)$ is the angle whose sine is $x$, it is equivalent to write

$$\sin(h(x)) = x.$$

Differentiating both sides of the previous equation, you have

$$\frac{d}{dx}[\sin(\mathrm{h}(\mathrm{x}))] = \frac{d}{dx}[\mathrm{x}],$$

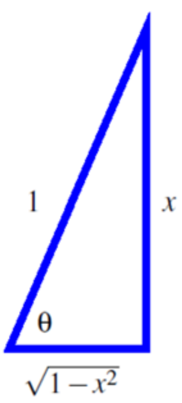
and by the fact that the right-hand side is simply 1 and by the chain rule applied to the left side,

$$\cos(h(x))h'(x) = 1.$$

Solving for $h'(x)$, it follows that

$$h'(x) = \frac{1}{\cos(h(x))}.$$

Finally, recall that $h(x) = \text{arcsin}\ (x)$, so the denominator of $h'(x)$ is the function $\cos\ (\text{arcsin}\ (x))$, or in other words, "the cosine of the angle whose sine is *x*." A bit of right triangle trigonometry allows you to simplify this expression considerably.




Let's say that $\theta = \text{arcsin(x)}$, so that $\theta$ is the angle whose sine is *x*.

Picture $\theta$ as an angle in a right triangle with hypotenuse 1, and a vertical leg of length *x*, as shown at left. The horizontal leg must be $\sqrt{1-x^2}$, by the Pythagorean Theorem.

Now, note particularly that $\theta\ = \text{arcsin(x)}$ since $\sin(\theta) = x$, and recall that you want to know a different expression for $\cos(\text{arcsin}(x))$. From the figure, $\cos(\text{arcsin}(x)) = \cos(\theta) = \sqrt{1-x^2}$.

Returning to your earlier work where you established that if $\text{h(x)} = \text{arcsin}(x)$, then

$h'(x) = \frac{1}{\cos(\text{arcsin}(x)}$ , you have

$$h'(x) = \frac{1}{\sqrt{1-x^2}}$$

**Inverse sine:** For all real numbers *x*, such that $-1 \leq x \leq 1$,

$$\frac{d}{dx}[\text{arcsin}(x)] = \frac{1}{\sqrt{1-x^2}}$$

**Investigation 3**: The following prompts in this activity will lead you to develop the derivative of the inverse tangent function.

3a) Let $r(x) = \text{arctan}(x)$. Use the relationship between the arctangent and tangent functions to rewrite this equation using only the tangent function.

b) Differentiate both sides of the equation you found in (a). Solve the resulting equation for $r\,'(x)$, writing $r\,'(x)$ as simply as possible in terms of a trigonometric function evaluated at $r(x)$.

c) Recall that $r\ (x) = \text{arctan}\ x$ . Update your expression for $r\,'(x)$ so that it only involves trigonometric functions and the independent variable $x$.

d) Introduce a right triangle with angle $\theta$ so that $\theta = \arctan(x)$. What are the three sides of the triangle?

e) In terms of only $x$ and 1, what is the value of $\cos(\arctan(x))$?

f) Use the results of your work above to find an expression involving only 1 and $x$ for $r'(x)$.

While derivatives for other inverse trigonometric functions can be established similarly, you primarily limit yourselves to the arcsine and arctangent functions. With these rules added to your library of derivatives of basic functions, you can differentiate even more functions using derivative shortcuts.

**Investigation 4**: Determine the derivative of each of the following functions.

a) $f(x) = x^3 \arctan(x) + e^x \ln(x)$

b) $p(t) = 2t^{\arcsin(t)}$

c) $h(z) = (\arcsin(5z) + \arctan(4 - z))^{27}$

d) $s(y) = \cot(\arctan(y))$

e) $m(v) = \ln(\sin^2(v) + 1)$

f) $g(w) = \arctan(\frac{\ln(w)}{1+w^2})$

## V. The link between the derivative of a function and the derivative of its inverse

Earlier, you saw an interesting relationship between the slopes of tangent lines to the natural exponential and natural logarithm functions at points that corresponded to reflection across the line y = x. That the two corresponding tangent lines having slopes that are reciprocals of one another is not a coincidence.

If you consider the general setting of a differentiable function f with differentiable inverse g such that y = f (x), if and only if x = g (y) , then you know that f (g(x) = x for every x in the domain of $f^{-1}$. Differentiating both sides of this equation with respect to $x$, you have

$$\frac{d}{dx}f\big(g(x)\big) = \frac{d}{dx}[x],$$

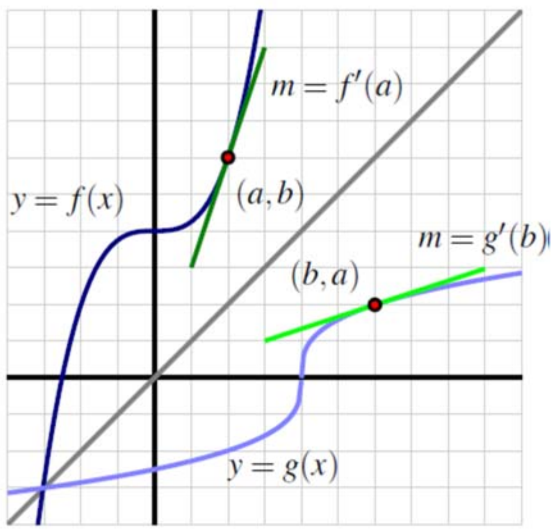
and by the chain rule,

$$f'(g(x))g'(x) = 1.$$

Solving for g'(x), you have

$$g'(x) = \frac{1}{f'g(x))}$$

Here you see that the slope of the tangent line to the inverse function $g$ at the point $(x, g(x))$ is precisely the reciprocal of the slope of the tangent line to the original function $f$ at the point $(g(x), f(g(x))) = (g(x), x)$.




To see this more clearly, consider the graph of the function $y = f(x)$ shown above, along with its inverse $y = g(x)$. Given a point $(a, b)$ that lies on the graph of $f$, you know that $(b, a)$ lies on the graph of $g$; said differently, $f(a) = b$ and $g(b) = a$. Now, applying the rule that $g'(x) = \frac{1}{f'g(x))}$ to the value $x = b$, you have

$$g'(b) = \frac{1}{f'g(x))} = \frac{1}{f'(a)}$$

which is precisely what you see in the figure: the slope of the tangent line to $g$ at $(b, a)$ is the reciprocal of the slope of the tangent line to $f$ at $(a, b)$, since these two lines are reflections of one another across the line $y = x$.

**Derivative of an inverse function**: Suppose that $f$ is a differentiable function with inverse $g$ and that $(a, b)$ is a point that lies on the graph of $f$ at which $f'(a) \neq 0$. Then

$$g'(b) = \frac{1}{f'(a)}$$

More generally, for any $x$ in the domain of $g'$, you have

$$g'(x) = \frac{1}{f'(g(x))}$$

The rules you derived for $\ln(x)$, $\arcsin(x)$, and $\arctan(x)$ are all just specific examples of this general property of the derivative of an inverse function. For example, with $g(x) = \ln(x)$ and $f(x) = e^x$, it follows that

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

## VI. Exercises

1. Determine the derivative of each of the following functions. Use proper notation and clearly identify the derivative rules you use.

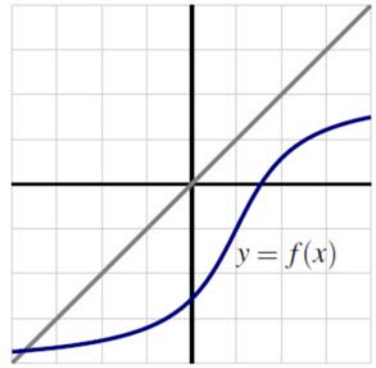
a) $f(x) = \ln(2\arctan(x) + 3\arcsin(x) + 5)$

b) $r(z) = \arctan(\ln(\arcsin(z)))$

c) $q(t) = \arctan^2(3t)\arcsin^4(7t)$

d) $g(v) = \ln\left(\frac{\arctan(v)}{\arcsin(v)+v^2}\right)$

2. Consider the graph of $y = f(x)$ provided at right and use it to answer the following questions:




a) Use the graph to estimate the value of $f'(1)$.

b) Sketch an approximate graph of $y = f^{-1}(x)$. Label at least three distinct points on the graph that correspond to three points on the graph of $f$.

c) Based on your work in (a), what is the value of $(f^{-1})'(-1)$? Why?

3. Let $f(x) = \frac{1}{4}x^3 + 4$

a) Sketch a graph of $y = f(x)$ and explain why $f$ is an invertible function.

b) Let $g$ be the inverse of $f$ and determine a formula for $g$.

c) Compute $f'(x)$, $g'(x)$, $f'(2)$, and $g'(6)$. What is the special relationship between
$f'(2)$ and $g'(6)$? Why?

4. Let $h(x) = x + \sin(x)$.

a) Sketch a graph of $y = h(x)$ and explain why $h$ must be invertible.

b) Explain why it does not appear to be algebraically possible to determine a formula for $h^{-1}$.

c) Observe that the point $(\frac{\pi}{2}, \frac{\pi}{2} + 1)$ lies on the graph of $y = h(x)$. Determine the value of $(h^{-1})'(\frac{\pi}{2}, \frac{\pi}{2} + 1)$

## VII. Practice – Khan Academy

1. Complete the following online practice exercises in Khan Academy's AP Calculus AB course:

a. https://www.khanacademy.org/math/ap-calculus-ab/differentiating-common-functions-ab/trigonometric-functions-differentiation-ab/e/differentiate-basic-trigonometric-functions
b. https://www.khanacademy.org/math/ap-calculus-ab/differentiating-common-functions-ab/trigonometric-functions-differentiation-ab/e/differentiate-trigonometric-functions
c. https://www.khanacademy.org/math/ap-calculus-ab/differentiating-common-functions-ab/exponential-functions-differentiation-ab/e/differentiate-exponential-functions-intro
d. https://www.khanacademy.org/math/ap-calculus-ab/differentiating-common-functions-ab/exponential-functions-differentiation-ab/e/differentiate-exponential-functions
e. https://www.khanacademy.org/math/ap-calculus-ab/differentiating-common-functions-ab/logarithmic-functions-differentiation-ab/e/differentiate-logarithmic-functions-intro
f. https://www.khanacademy.org/math/ap-calculus-ab/differentiating-common-functions-ab/logarithmic-functions-differentiation-ab/e/differentiate-logarithmic-functions
g. https://www.khanacademy.org/math/ap-calculus-ab/advanced-differentiation-ab/derivatives-of-inverse-functions-ab/e/derivatives-of-inverse-functions

2. Optional: none