2.2H L'Hôpital to the Rescue

Using Derivatives to Evaluate Limits

Because differential calculus is based on the definition of the derivative, and the definition of the derivative involves a limit, there is a sense in which all of calculus rests on limits. In addition, the limit involved in the limit definition of the derivative is one that always generates an indeterminate form of $\frac{0}{0}$. If f is a differentiable function for which $f'(x)$ exists, then when you consider

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

it follows that not only does $h \to 0$ in the denominator, but also $f(a+h) - f(a) \to 0$ in the numerator, since f is continuous. Thus, the fundamental form of the limit involved in the definition of $f'(x)$ is $\frac{0}{0}$. Remember, saying a limit has an ***indeterminate form*** only means that you don't yet know its value and have more work to do: indeed, limits of the form $\frac{0}{0}$ can take on any value, as is evidenced by evaluating $f'(x)$ for varying values of x for a function such as $f'(x) = x^2$.

Of course, you have learned many different techniques for evaluating the limits that result from the derivative definition, and including many shortcut rules that enable you to evaluate these limits quickly and easily. In this lesson, you turn the situation upside-down: rather than using limits to evaluate derivatives, you explore how to use derivatives to evaluate certain limits. This topic will combine several different ideas, including limits, derivative shortcuts, local linearity, and the tangent line approximation.

Investigation 1: Let h be the function given by $h(x) = \frac{x^5+x-2}{x^2-1}$.

a) What is the domain of h?

b) Explain why $\lim_{x \to 1} \frac{x^5 + x - 2}{x^2 - 1}$ results in an indeterminate form.

c) Next, investigate the behavior of both the numerator and denominator of h near the point where $x = 1$. Let $f(x) = x^5 + x\ \ 2$ and $g(x) = x^2 - 1$. Find the local linearizations of f and g at $a = 1$, and call these functions $L_f(x)$ and $L_g(x)$, respectively.

d) Explain why $h(x) \approx \frac{L_f(x)}{L_g(x)}$ for x near $a = 1$.

e) Using your work from (c) and (d), evaluate

$$\lim_{x \to 1} \frac{L_f(x)}{L_g(x)}$$

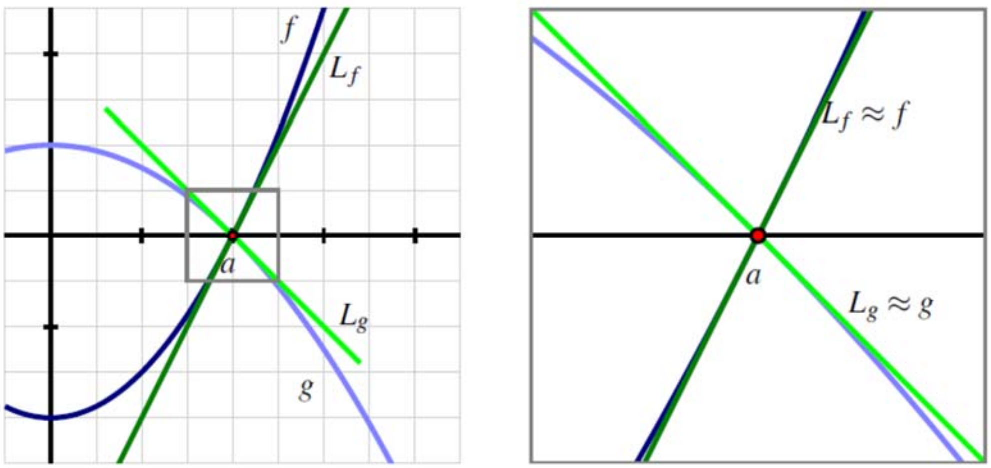
What do you think your result tells you about $\lim_{x \to 1} h(x)$?

f) Investigate the function $h(x)$ graphically and numerically near $x = 1$. What do you think is the value of $\lim_{x \to 1} h(x)$?

II. Using derivatives to evaluate indeterminate limits of the form $\frac{0}{0}$

The fundamental idea of Investigation 1 – that you can evaluate an indeterminate limit of the form $\frac{0}{0}$ by replacing each of the numerator and denominator with their local linearizations at the point of interest – can be generalized in a way that enables you to easily evaluate a wide range of limits. Assume that you have a function $h(x)$ that can be written in the form $h(x) = \frac{f(x)}{g(x)}$ where f and g are both differentiable at $x = a$ and for which $f(a) = g(a) = 0$. You are interested in finding a way to evaluate the indeterminate limit given by $\lim_{x \to a} h(x)$.

In the figure below, you see a visual representation of the situation involving such functions f and g. In particular, notice that both f and g have an x-intercept at the point where $x = a$. Since each function is differentiable, each is locally linear, and you can find their respective tangent line approximations L_f and L_g at $x = a$, which are also shown in the figure. Since you are interested in the limit of $\frac{f(x)}{g(x)}$ as $x \to a$, the individual behaviors of $f(x)$ and $g(x)$ as $x \to a$ are key to understand. Here, you take advantage of the fact that each function and its tangent line approximation become indistinguishable as $x \to a$.



First, recall that $L_f(x) = f'(a)(x - a) + f(a)$ and $L_g(x) = g'(a)(x - a) + g(a)$. The critical observation is that when taking the limit, because x is getting arbitrarily close to a, you can replace f with L_f and replace g with L_g, and thus you observe that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{L_f(x)}{L_g(x)}$$

$$= \lim_{x \to a} \frac{f'(a)(x - a) + f(a)}{g'(a)(x - a) + g(a)}$$

Next, remember a key fundamental assumption: that both $f(a) = 0$ and $g(a) = 0$, as this is precisely what makes the original limit indeterminate. Substituting these values for $f(a)$ and $g(a)$ in the limit above, you now have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x - a)}{g'(a)(x - a)}$$

$$= \lim_{x \to a} \frac{f'(a)}{g'(a)}$$

where the latter equality holds since x is approaching (but not equal to) a, so $\frac{x-a}{x-a} = 1$. Finally, notice that $f'(a)$ is constant with respect to x, and thus

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

You have, of course, implicitly made the assumption that $g'(a) \neq 0$, which is essential to the overall limit having the value $\frac{f'(a)}{g'(a)}$. Let's summarize the work above with the statement of L'Hôpital's Rule, which is the formal name of the result you have shown.

L'Hôpital's Rule: Let f and g be differentiable at $x = a$, and suppose that $f(a) = g(a) = 0$, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

In practice, you will typically work with a slightly more general version of L'Hôpital's Rule, which states that (under the identical assumptions as the boxed rule above and the extra assumption that g' is continuous at $x = a$)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

provided the right-hand limit exists. This form reflects the fundamental benefit of L'Hôpital's Rule: if $\frac{f(x)}{g(x)}$ produces an indeterminate limit of form $\frac{0}{0}$ as $x \to a$, it is equivalent to consider the limit of the quotient of the two functions' derivatives, $\frac{f'(x)}{g'(x)}$. For example, if you consider the limit from Investigation 1,

$$\lim_{x \to 1} \frac{x^5 + x - 2}{x^2 - 1},$$

by L'Hôpital's Rule you have that

$$\lim_{x \to 1} \frac{x^5 + x - 2}{x^2 - 1} = \lim_{x \to 1} \frac{5x^4 + 1}{2x} = \frac{6}{2} = 3.$$

By being able to replace the numerator and denominator with their respective derivatives, you often move from an indeterminate limit to one whose value you can easily determine.

Investigation 2: Evaluate each of the following limits. If you use L'Hôpital's Rule, indicate where it was used, and be certain its hypotheses are met before you apply it.

a) $\lim_{x \to 0} \frac{\ln(1+x)}{x}$

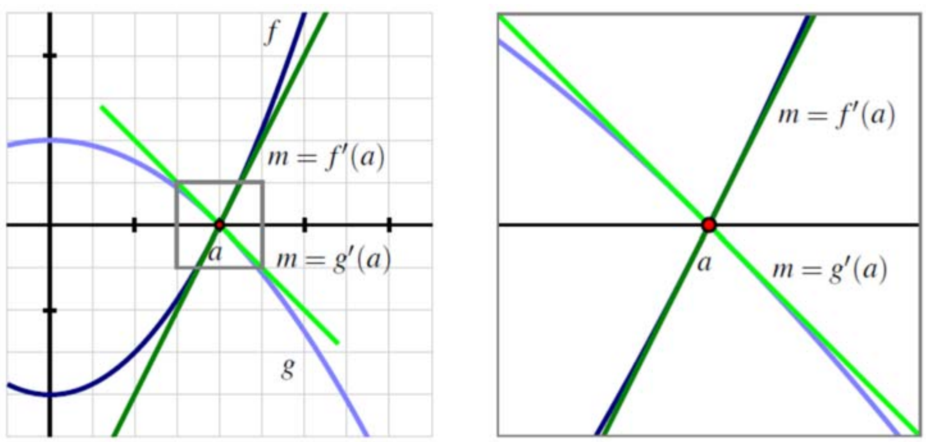
b) $\lim_{x \to \pi} \frac{\cos(x)}{x}$

c) $\lim_{x \to 1} \frac{2\ln(x)}{1-e^{x-1}}$

d) $\lim_{x \to 0} \frac{\sin(x)-x}{\cos(2x)-1}$

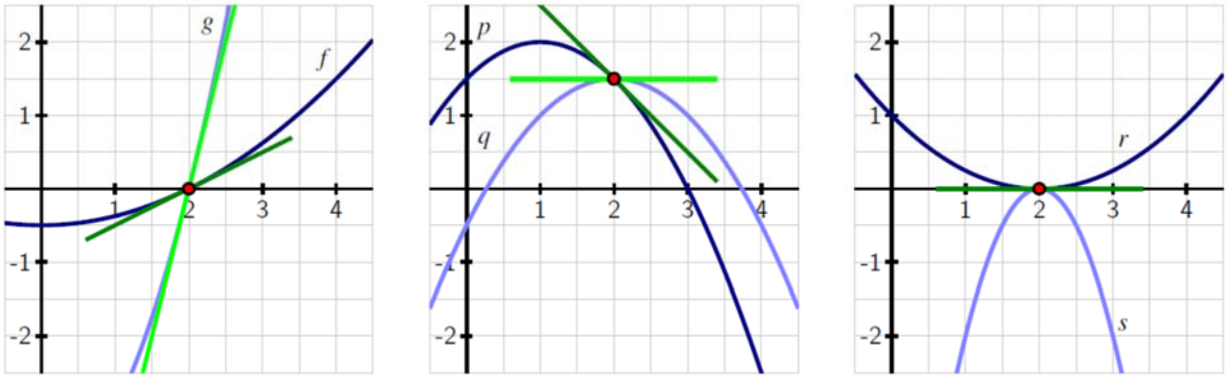
While L'Hôpital's Rule can be applied in an entirely algebraic way, it is important to remember that the genesis of the rule is graphical: the main idea is that the slopes of the tangent lines to f and g at $x \to a$ determine the value of the limit of $\frac{f(x)}{g(x)}$ as $x \to a$. You see this in the figure below, which is a modified version of graphs used earlier, you can see from the grid that f '(a) = 2 and g'(a) = -1, hence by L'Hôpital's Rule,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \frac{2}{-1} = -2.$$



Notice that it's not the fact that f and g both approach zero that matters most, but rather the rate at which each approaches zero that determines the value of the limit. This is a good way to remember what L'Hôpital's Rule says: if $f(a) = g(a) = 0$, the limit of $\frac{f(x)}{g(x)}$ as $x \to a$ is given by the ratio of the slopes of f and g at $x = a$.

Investigation 3: In this activity, you reason graphically from the following figure to evaluate limits of ratios of functions about which some information is known.



a) Use the left-hand graph to determine the values of $f(2)$, $f'(2)$, $g(2)$, and $g'(2)$. Then, evaluate

$$\lim_{x\to 2}\frac{f(x)}{g(x)}$$

(b) Use the middle graph to find $p(2)$, $p'(2)$, $q(2)$, and $q'(2)$. Then, determine the value of

$$\lim_{x\to 2}\frac{p(x)}{q(x)}$$

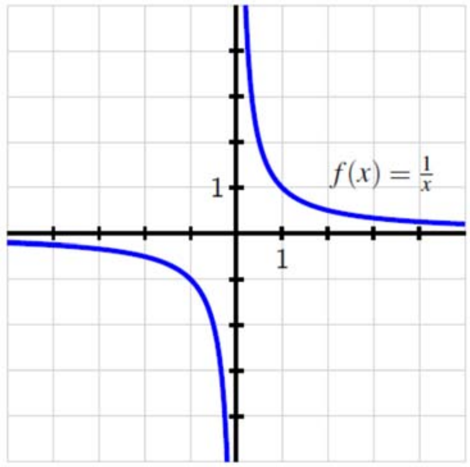
c) Use the right-hand graph to compute $r(2)$, $r'(2)$, $s(2)$, $s'(2)$. Explain why you cannot determine the exact value of

$$\lim_{x\to 2}\frac{r(x)}{s(x)}$$

without further information being provided, but that you can determine the sign of $\lim_{x\to 2}\frac{r(x)}{s(x)}$. In addition, state what the sign of the limit will be, with justification.

III. Limits involving ∞

The concept of infinity, denoted ∞, arises naturally in calculus, like it does in much of mathematics. It is important to note from the outset that ∞ is a concept, but not a number itself. Indeed, the notion of ∞ naturally invokes the idea of limits. Consider, for example, the function $f(x) = \frac{1}{x}$, whose graph is pictured below.



Notice that $x = 0$ is not in the domain of f, so you may naturally wonder what happens as $x \to 0$. As $x \to 0^+$, you observe that f (x) ***increases without bound***. That is, you can make the value of $f(x)$ as large as you like by taking x closer and closer (but not equal) to 0, while keeping $x > 0$. This is a good way to think about what infinity represents: a quantity is tending to infinity if there is no single number that the quantity is always less than.

Recall that when you write $\lim_{x \to a} f(x) = L$, you can make $f(x)$ as close to L as you'd like by taking x sufficiently close (but not equal) to a. You thus expand this notation and language to include the possibility that either L or a can be ∞[1]. For instance, for $f(x) = \frac{1}{x}$, you now write

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

By which, it is meant that one can make $\frac{1}{x}$ as large as you like by taking x sufficiently close to (but not equal) to 0. In a similar way,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

since you can make 1 as close to 0 as you'd like by taking x sufficiently large (i.e., by letting x increase without bound).

[1] The College Board does not condone this terminology for the AP test; instead they prefer saying this limit is nonexistent.

In general, you understand the notation $\lim_{x \to a} f(x) = \infty$ to mean that you can make $f(x)$ as large as you'd like by taking x sufficiently close (but not equal) to a, and the notation $\lim_{x \to \infty} f(x) = L$ to mean that you can make $f(x)$ as close to L as you'd like by taking x sufficiently large. This notation applies to left- and right-hand limits, plus you can also use limits involving $-\infty$. For example, returning to the graph $f(x) = \frac{1}{x}$ on the previous page, one can say that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \text{ and } \lim_{x \to \infty} \frac{1}{x} = 0$$

Finally, you write

$$\lim_{x \to \infty} f(x) = \infty$$

when you can make the value of $f(x)$ as large as you'd like by taking x sufficiently large. For example,

$$\lim_{x \to \infty} x^2 = \infty$$

Note particularly that limits involving infinity identify vertical and horizontal asymptotes of a function. If $\lim_{x \to a} f(x) = \infty$, then $x = a$ is a vertical asymptote of f, while if $\lim_{x \to \infty} f(x) = L$, then y = L is a horizontal asymptote of f. Similar statements can be made using $-\infty$, as well as with left- and right-hand limits as $x \to a^+$ or $x \to a^+$.

In precalculus classes, it is common to study the *end behavior* of certain families of functions, by which you mean the behavior of a function as $x \to \infty$ and as $x \to -\infty$.

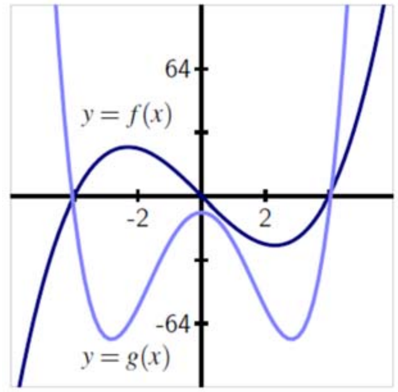
Next, you will examine a library of some familiar functions and note the values of several limits involving ∞.

- For the natural exponential function e^x, notice that $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to -\infty} e^x = 0$, while for the related exponential decay function e^{-x} , observe that these limits are reversed, with $\lim_{x \to \infty} e^{-x} = 0$ and $\lim_{x \to -\infty} e^{-x} = \infty$.

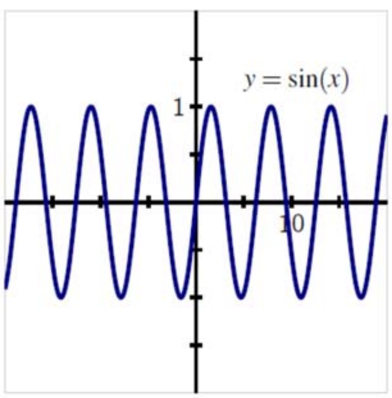
  Turning to the natural logarithm function, the $\lim_{x \to 0^+} \ln(x) = -\infty$ and $\lim_{x \to \infty} \ln(x) = \infty$. While both e^x and $\ln(x)$ grow without bound as $x \to \infty$, the exponential function does so much more quickly than the logarithm function does.

You'll soon use limits to quantify what is meant by "quickly."

- For polynomial functions of the form $p(x) = a_n x^n + a_{n-1} x^{n+1} + \cdots + a_1 x + a_0$, the end behavior depends on the sign of a_n and whether the highest power n is even or odd.



  - If n is even and an is positive, then $\lim_{x\to\infty} \text{p(x)} = \infty$ and $\lim_{x\to-\infty} \text{p(x)} = \infty$, as in the plot of g at right.
  - If instead a_n is negative, then $\lim_{x\to\infty} \text{p(x)} = -\infty$ and $\lim_{x\to-\infty} \text{p(x)} = -\infty$.
  - In the situation where n is odd, then either $\lim_{x\to\infty} \text{p(x)} = \infty$ and $\lim_{x\to-\infty} \text{p(x)} = -\infty$ (which occurs when a_n is positive, as in the graph of f in Figure 2.23), or $\lim_{x\to\infty} \text{p(x)} = -\infty$ and $\lim_{x\to-\infty} \text{p(x)} = \infty$ (when a_n is negative).



A function can fail to have a limit as $x \to \infty$. For example, consider the plot of the sine function at left. Because the function continues oscillating between -1 and 1 as $x \to \infty$, you say that $\lim_{x\to\infty} \sin(\text{x})$ does not exist.

- Finally, it is straightforward to analyze the behavior of any rational function as $x \to \infty$.

  Consider, for example, the function $q(x) = \frac{3x^2-4x+5}{7x^2+9x-10}$.

Note that both $(3x^2 - 4x + 5) \to \infty$ as $x \to \infty$ and $(7x^2 + 9x - 10) \to \infty$ as $x \to \infty$.

Here you say that $q(x)$ has indeterminate form $\frac{\infty}{\infty}$, much like you did when you encountered limits of the form $\frac{0}{0}$. You can determine the value of this limit through a standard algebraic approach. Multiplying the numerator and denominator each by $\frac{1}{x^2}$, you find that

$$\lim_{x\to\infty} q(x) = \lim_{x\to\infty} \frac{3x^2-4x+5}{7x^2+9x-10} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\lim_{x\to\infty} q(x) = \lim_{x\to\infty} \frac{3-4\cdot\frac{1}{x}+5\cdot\frac{1}{x^2}}{7+9\cdot\frac{1}{x}-10\cdot\frac{1}{x^2}} = \frac{3}{7}$$

since $\frac{1}{x^2} \to 0$ and $\frac{1}{x} \to 0$ as $x \to \infty$. This shows that the rational function q has a horizontal asymptote at $y = 3$. A similar approach can be used to determine the limit of any rational function as $x \to -\infty$.

But how should you handle a limit such as

$$\lim_{x\to\infty} \frac{x^2}{e^x} \ ?$$

Here, both $x^2 \to \infty$ and $e^x \to \infty$, but there is not an obvious algebraic approach that enables you to find the limit's value. Fortunately, it turns out that L'Hôpital's Rule extends to cases involving infinity.

L'Hôpital's Rule (∞): If f and g are differentiable and both 0 or both approach $\pm\infty$ as $x \to a$, (where a is allowed to be ∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)}{g'(a)}$$

(To be technically correct, you need to the additional hypothesis that $g'(x) \neq 0$ on an open interval that contains a or in every neighborhood of infinity if a is ∞; this is almost always met in practice.)

To evaluate $\lim_{x\to\infty}\frac{x^2}{e^x}$, you first must confirm that you can apply L'Hôpital's Rule. Write something like:

Since both $x^2 \to \infty$ and $e^x \to \infty$ as $x \to \infty$, the indeterminate form $\frac{\infty}{\infty}$ results.

$\therefore$[2] L'Hôpital's Rule is applied, as follows:

$$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x}$$

This updated limit is still indeterminate and of the form $\frac{\infty}{\infty}$

$\therefore$ L'Hôpital's Rule is applied a second time, as follows:

$$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x}$$

Since 2 is constant and $e^x \to \infty$ as $x \to \infty$,, it follows that $\frac{2}{e^x}x \to \infty$

This shows that

$$\lim_{x\to\infty}\frac{x^2}{e^x} = 0$$

Investigation 4: Evaluate each of the following limits. If you use L'Hôpital's Rule, indicate where it was used, and verify its hypotheses are met before you apply it.

a) $\lim_{x\to\infty}\frac{x}{\ln(x)}$

b) $\lim_{x\to\infty}\frac{e^x+x}{2e^x+x^2}$

c) $\lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x}}$

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[2] This is the abbreviation for "therefore," or "in conclusion."

d) $\lim_{x\to\frac{\pi}{2}^-} \frac{\tan(x)}{x-\frac{\pi}{2}}$

(e) $\lim_{x\to\infty} xe^{-x}$

When you are considering the limit of a quotient of two functions $\frac{f(x)}{g(x)}$ that results in an indeterminate form of $\frac{\infty}{\infty}$, in essence you are asking which function is growing faster without bound. You say that the function g ***dominates*** the function *f* as $x \to \infty$ provided that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

whereas *f* dominates g provided that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$. Finally, if the value of $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is finite and nonzero, you say that *f* and g grow at the same rate.

IV. Exercises

1. Let *f* and g be differentiable functions about which the following information is known:

f $(3) = g(3) = 0, f'(3) = g'(3) = 0, f''(3) = ?2$, and $g''(3) = 1$.

Let a new function h be given by the rule $h(x) = \frac{f(x)}{g(x)}$. On the same set of axes, sketch possible graphs of *f* and g near $x = 3$, and use the provided information to determine the value of

$$\lim_{x\to 3} \frac{f(x)}{g(x)}$$

Provide explanation to support your conclusion.

2. Find all vertical and horizontal asymptotes of the function

$$R(x) = \frac{3(x-a)(x-b)}{5(x-a)(x-c)}$$

where a, b, and c are distinct, arbitrary constants. In addition, state all values of x for which R is not continuous. Sketch a possible graph of R, clearly labeling the values of a, b, and c.

3. Consider the function $g(x) = x^{2x}$, which is defined for all $x > 0$. Observe that $\lim_{x\to 0^+} g(x) = 0$ is indeterminate due to its form of 0^0. (Think about how you know that $0^k = 0$ for all $k > 0$, while $b^0 = 1$ for all $b \neq 0$, but that neither rule can apply to 0^0.)

a) Let $h(x) = \ln(g(x))$. Explain why $h(x) = 2x\ln(x)$.

b) Next, explain why it is equivalent to write $h(x) = \frac{2\ln(x)}{\frac{1}{x}}$.

c) Use L'Hôpital's Rule and your work in (b) to compute $\lim_{x\to 0^+} h(x)$.

d) Based on the value of $\lim_{x\to 0^+} h(x)$, determine $\lim_{x\to 0^+} g(x)$.

4. Recall you say that function g dominates function f provided that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

a) Which function dominates the other: ln(x) or $\sqrt{x}$?

b) Which function dominates the other: ln (x) or $\sqrt[n]{x}$? (n can be any positive integer)

c) Explain why e^x will dominate any polynomial function.

d) Explain why x^n will dominate $\ln(x)$ for any positive integer n.

e) Give any example of two nonlinear functions such that neither dominates the other.

V. Practice – Khan Academy

1. Complete the following practice exercises in the Derivative Applications unit of Khan Academy's AP Calculus AB course:
   a. https://www.khanacademy.org/math/ap-calculus-ab/derivative-applications-ab/lhopitals-rule-ab/e/lhopitals_rule
   b. https://www.khanacademy.org/math/ap-calculus-ab/derivative-applications-ab/lhopitals-rule-ab/e/lhopitals-rule-composite-exponential-functions