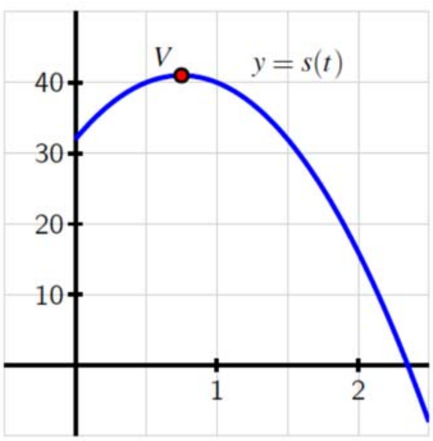
2.3A Mind Your Behavior

Using Derivatives to Identify Extreme Values

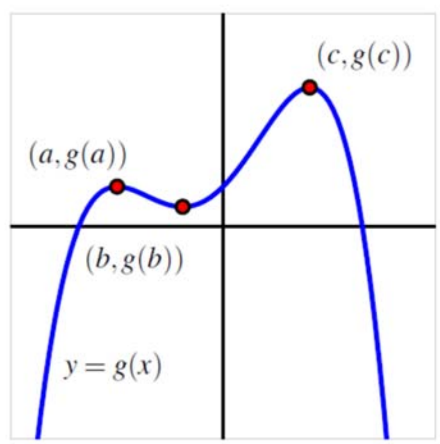
In many different settings, you are interested in knowing where a function achieves its least and greatest values. These can be important in applications – say to identify a point at which maximum profit or minimum cost occurs – or in theory to understand how to characterize the behavior of a function or a family of related functions.

Consider the simple and familiar example of a parabolic function such as

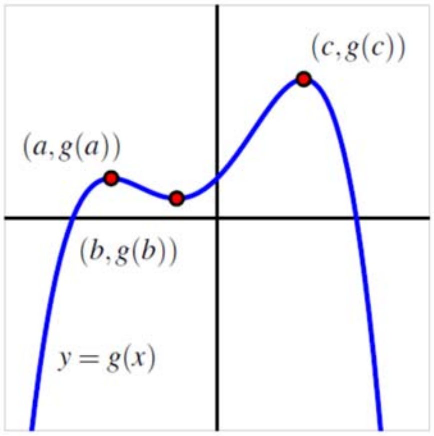


$s(t) = 16t^2 + 32t + 48$ (shown at left) that represents the height of an object tossed vertically: its maximum value occurs at the vertex of the parabola and represents the highest value that the object reaches. Moreover, this maximum value identifies an especially important point on the graph, the point at which the curve changes from increasing to decreasing.

More generally, for any function you consider, you can investigate where its lowest and highest points occur in comparison to points nearby or to all possible points on the graph. Given a function f, you say that $f(c)$ is a ***global*** or ***absolute maximum*** provided that $f(c) \geq f(x)$ for all x in the domain of f, and similarly call $f(c)$ a ***global*** or ***absolute minimum*** whenever $f(c) \leq f(x)$ for all x in the domain of f.



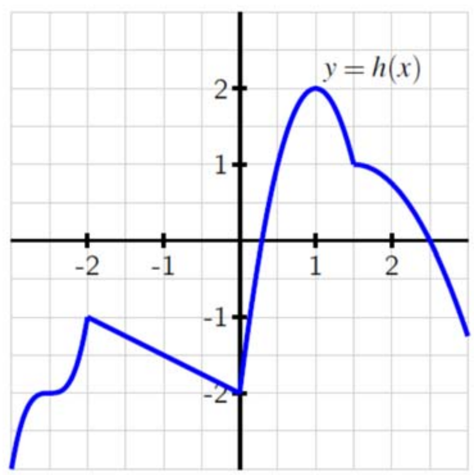
For instance, for the function g given at right, g has a global maximum of $g(c)$, but g does not appear to have a global minimum, as the graph of g seems to decrease without bound. You note that the point $(c, g(c))$ marks a fundamental change in the behavior of g, where g changes from increasing to decreasing; similar things happen at both $(a, g\,(a))$ and$(b, g(b))$, although these points are not global mins or maxes.



For any function f, you say that $f\,(c)$ is a ***local maximum*** or ***relative maximum*** provided that $f(c) \geq f(x)$ for all x near c, while $f\,(c)$ is called a ***local*** or ***relative minimum*** whenever $f(c) \leq f(x)$ for all x near c. Any maximum or minimum may be called an ***extreme value*** of f . For example, in the graph at left, g has a relative minimum of $g(b)$ at the point $(b, g(b))$ and a relative maximum of $g(a)$ at $(a, g(a))$. You have already identified the global maximum of g as $g(c)$; this global maximum can also be considered a relative maximum.

You would like to use fundamental calculus ideas to help you identify and classify key function behavior, including the location of relative extremes. Of course, if you are given a graph of a function, it is often straightforward to locate these important behaviors visually. You investigate this situation in the first investigation.

Investigation 1: Consider the function h given by the graph below right. Use the graph to answer each of the following questions.



a) Identify all of the values of c for which $h(c)$ is a local maximum of h.

b) Identify all of the values of c for which $h(c)$ is a local minimum of h.

c) Does h have a global maximum on the interval [-3, 3]? If so, what is the value of this global maximum?

d) Does h have a global minimum on the interval [-3, 3]? If so, what is its value?

e) Identify all values of c for which $h'(c) = 0$.

f) Identify all values of c for which $h'(c)$ does not exist.

g) True or false: every relative maximum and minimum of h occurs at a point where

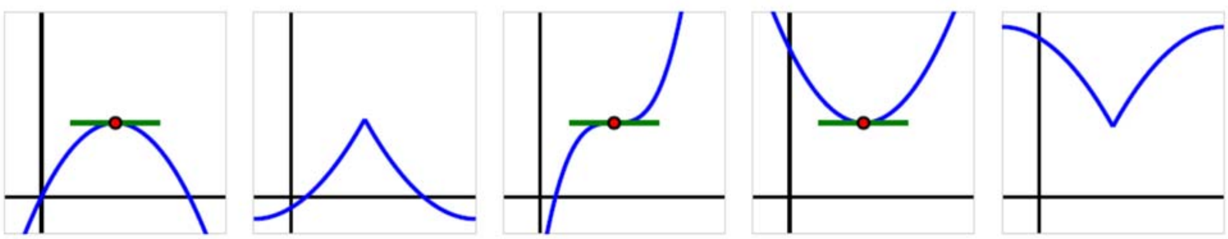
$h'(c)$ is either zero or does not exist.

h) True or false: at every point where $h'(c)$ is zero or does not exist, h has a relative maximum or minimum.

II. Critical numbers and the first derivative test

If a function has a relative extreme value at a point $(c, f(c))$, the function must change its behavior at c regarding whether it is increasing or decreasing before or after the point.

For example, if a continuous function has a relative maximum at c, such as those pictured in the two leftmost functions in the figure below, then it is both necessary and sufficient that the function change from being increasing just before c to decreasing just after c. In the same way, a continuous function has a relative minimum at c if and only if the function changes from decreasing to increasing at c. See, for instance, the two functions pictured below right. There are only two possible ways for these changes in behavior to occur: either $f'(c) = 0$ or $f'(c)$ is undefined.



Because these values of c are so important, you call them ***critical numbers***. More specifically, you say that a function f has a critical number at $x = c$ provided that c is in the domain of f , and $f'(c) = 0$ or $f'(c)$ is undefined. Critical numbers provide you with the only possible locations where the function f may have relative extremes. Note that not every critical number produces a maximum or minimum; in the middle graph above, the function pictured there has a horizontal tangent line at the noted point, but the function is increasing before and increasing after, so the critical number does not yield a location where the function is greater than every value nearby, nor less than every value nearby.

You also sometimes use the terminology that, when c is a critical number, that $(c, f(c))$ is a critical point of the function, or that $f(c)$ is a critical value .

The ***first derivative test*** summarizes how sign changes in the first derivative indicate the presence of a local maximum or minimum for a given function.

> **First Derivative Test:** If p is a critical number of a continuous function f that is differentiable near p (except possibly a $x = p$), then f has a relative maximum at p if and only if f' changes sign from positive to negative at p, and f has a relative minimum at p if and only if f' changes sign from negative to positive at p.

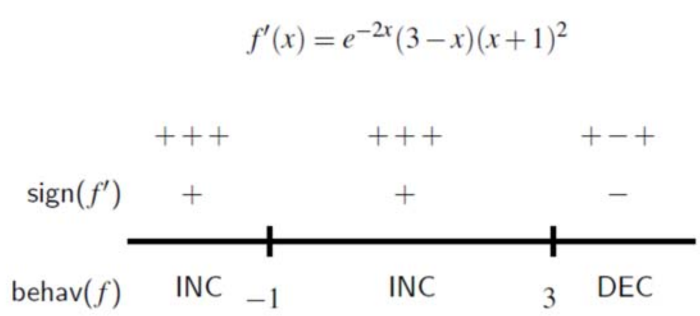
Example 1. Let f be a function whose derivative is given by the formula $f'(x) = e^{-2x}(3-x)(x+1)^2$. Determine all critical numbers of f and decide whether a relative maximum, relative minimum, or neither occurs at each.

Solution. Since you already have $f'(x)$ written in factored form, it is straightforward to find the critical numbers of f . Since $f'(x)$ is defined for all values of x, you need only determine where $f'(x) = 0$. From the equation

$$e^{-2x}(3-x)(x+1)^2 = 0$$

and the zero product property, it follows that $x = 3$ and $x = 1$ are critical numbers of f. (Note particularly that there is no value of x that makes $e^{-2x} = 0$.)

Next, to apply the first derivative test, you'd like to know the sign of $f'(x)$ at inputs near the critical numbers. Because the critical numbers are the only locations at which f' can change sign, it follows that the sign of the derivative is the same on each of the intervals created by the critical numbers: for instance, the sign of f' must be the same for every $x <$ -1. You create a first derivative ***sign chart***[1] to summarize the sign of f' on the relevant intervals along with the corresponding behavior of f.



[1] A hand drawn sign chart is a required element for free response questions involving behavior on the AP tests

The first derivative sign chart on the previous page comes from thinking about the sign of each of the terms in the factored form of $f(x)$ at one selected point in the interval under consideration. For instance, for $x < 1$, you could consider $x = 2$ and determine the sign of $e^{-2x}(3 - x)(x + 1)^2$ at the value $x = 2$. You note that both e^{-2x} and $(x + 1)^2$ are positive regardless of the value of x, while $(3 - x)$ is also positive at $x = 2$. Hence, each of the three terms in f' is positive, which you indicate by writing "$+ + +$." Taking the product of three positive terms obviously results in a value that is positive, which you denote by the "$+$" in the interval to the left of $x = 1$ indicating the overall sign of f'. And, since f' is positive on that interval, you further know that f is increasing, which you summarize by writing "INC" to represent the corresponding behavior of f. In a similar way, you find that f' is positive and f is increasing on $1 < x < 3$, and f' is negative and f is decreasing for $x > 3$.

Now, by the first derivative test, to find relative extremes of f you look for critical numbers at which f' changes sign. In this example, f' only changes sign at $x = 3$, where f' changes from positive to negative, and thus f has a relative maximum at $x = 3$. While f has a critical number at $x = -1$, since f is increasing both before and after $x = -1$, f has neither a minimum nor a maximum at $x = -1$.

Investigation 2: Suppose that $g(x)$ is a function continuous for every value of $x \neq 2$ whose first derivative is $g'(x) = \frac{(x+4)(x-1)^2}{x-2}$. Further, assume that it is known that g has a vertical asymptote at $x = 2$.

a) Determine all critical numbers of g.

b) By developing a carefully labeled first derivative sign chart, decide whether g has as a local maximum, local minimum, or neither at each critical number.

c) Does g have a global maximum? global minimum? Justify your claims.

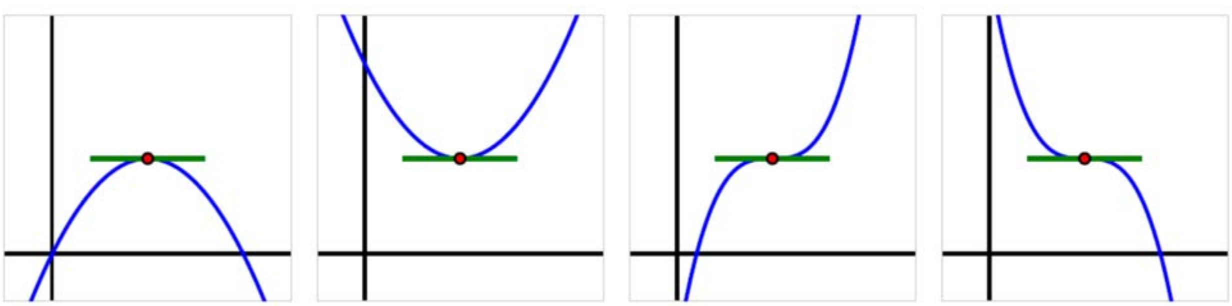
d) What is the value of $\lim_{x \to \infty} g'(x)$? What does the value of this limit tell you about the long-term behavior of g?

(e) Sketch a possible graph of $y = g(x)$.

III. The second derivative test

Recall that the second derivative of a function tells you several important things about the behavior of the function itself. For instance, if f'' is positive on an interval, then you know that f' is increasing on that interval and, consequently, that f is concave up, which also tells you that throughout the interval the tangent line to $y = f(x)$ lies below the curve at every point. In this situation where you know that $f'(p) = 0$, it turns out that the sign of the second derivative determines whether f has a local minimum or local maximum at the critical number p.

In the figure below, you see the four possibilities for a function f that has a critical number p at which $f'(p) = 0$, provided f ,, p is not zero on an interval including p (except possibly at p). On either side of the critical number, f''can be either positive or negative, and hence f can be either concave up or concave down. In the first two graphs, f does not change concavity at p, and in those situations, f has either a local minimum or local maximum. In particular, if $f'(p) = 0$ and $f''(p) < 0$, then you know f is concave down at p with a horizontal tangent line, and this guarantees f has a local maximum there. This fact, along with the corresponding statement for when $f''(p)$ is positive, is stated in the second derivative test.



Second Derivative Test: If p is a critical number of a continuous function f such that such that $f'(p) = 0$ and $f''(p) \neq 0$, then f has a relative maximum at p if and only $f''(p) < 0$, and f has a relative minimum at p if and only if $f''(p) < 0$.

In the event that $f''(p) = 0$, the second derivative test is inconclusive. That is, the test doesn't provide you any information. This is because if $f''(p) = 0$, it is possible that f has a local minimum, local maximum, or neither.[2]

Just as a first derivative sign chart reveals all of the increasing and decreasing behavior of a function, you can construct a second derivative sign chart that demonstrates all of the important information involving concavity.

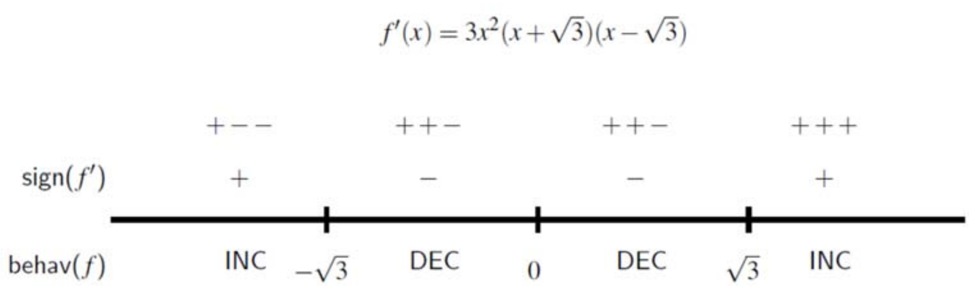
Example 2. Let $f(x)$ be a function whose first derivative is $f'(x) = 3x^4 - 9x^2$. Construct both first and second derivative sign charts for f, fully discuss where f is increasing and decreasing and concave up and concave down, identify all relative extreme values, and sketch a possible graph of f.

Solution. Since you know $f'(x) = 3x^4 - 9x^2$, you can find the critical numbers of f by solving $f'(x) = 3x^4 - 9x^2 = 0$. Factoring, you observe that

$$0 = 3x^2(x^2 - 3)$$

$$0 = 3x^2(x + \sqrt{3})(x - \sqrt{3})$$

so that $x = 0,\ \pm\sqrt{3}$ are the three critical numbers of f. It then follows that the first derivative sign chart for f is given below.



[2] Consider the functions f (x) = x^4, g(x) = -x4, and h(x) = x^3 at the critical point p = 0.

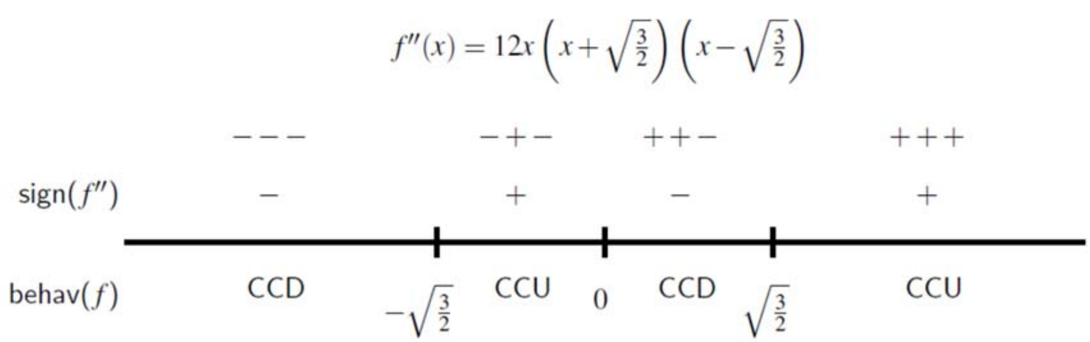
Thus, f is increasing on the intervals $(-\infty, -\sqrt{3})$ and $(\infty, \sqrt{3})$, while f is decreasing on $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$. Note particularly that by the first derivative test, this information tells you that f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$. While f also has a critical number at $x = 0$, neither a maximum nor minimum occurs there since f' does not change sign at $x = 0$.

Next, you move on to investigate concavity. Differentiating $f'(x) = 3x^4 - 9x^2$, you see that $f''(x) = 12x^3 - 18x$. Since you are interested in knowing the intervals on which f'' is positive and negative, you first find where $f'' = 0$. Observe that

$$0 = 12x^3 - 18x \ = 12x\left(x^2 - \frac{3}{2}\right)$$

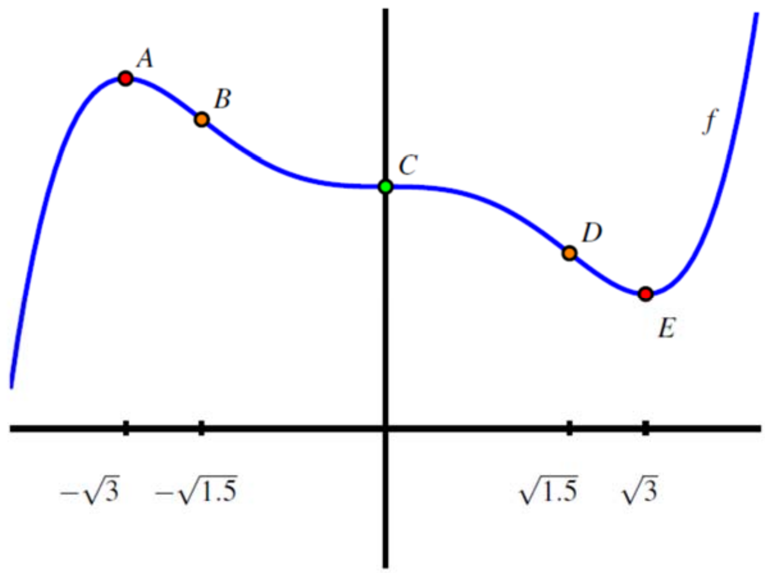
$$12x\left(x + \sqrt{\frac{3}{2}}\right)\left(x - \sqrt{\frac{3}{2}}\right)$$

which implies that $x = 0, \pm\sqrt{\frac{3}{2}}$. Building a sign chart for f'' in the exact same way you do for f', you see the result shown below.



Therefore, f is concave down on the intervals $(-\infty, -\sqrt{\frac{3}{2}})$ and $(0, \sqrt{\frac{3}{2}})$, and concave up on $(-\sqrt{\frac{3}{2}}, 0)$ and $(\sqrt{\frac{3}{2}}, \infty)$.

Putting all of the above information together, you can complete an accurate possible graph of *f* like the one pictured below. Notice that all of the critical points have been clearly identified.

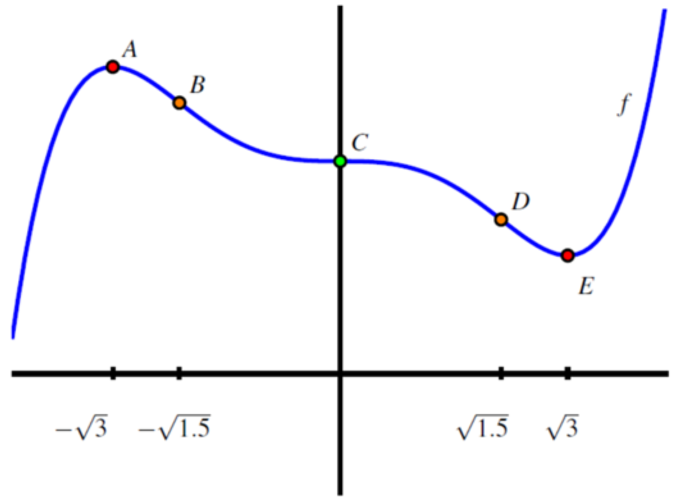


Explanation[3]: The point $A = (\sqrt{3}, f(\sqrt{3}))$ is a local maximum, as *f* is increasing prior to *A* and decreasing after; similarly, the point $E = (-\sqrt{3}, f(-\sqrt{3}))$ is a local minimum. *f* is concave down at *A* and concave up at *B*, which is consistent both with the second derivative sign chart and the second derivative test. At points *B* and *D*, concavity changes, as seen in the results of the second derivative sign chart. Finally, at point *C*'(*f*) has a critical point with a horizontal tangent line, but neither a maximum nor a minimum occurs there since *f* is decreasing both before and after *C*. It is also the case that concavity changes at *C*.

While you completely understand where *f* is increasing and decreasing, where *f* is concave up and concave down, and where *f* has relative extremes, you do not know any specific information about the *y*-coordinates of points on the curve. For instance, while you know that f has a local maximum at $x = \sqrt{3}$, you don't know the value of that maximum because you do not know $f(-\sqrt{3}$. Any vertical translation of your sketch of *f* in the figure above would satisfy the given criteria for *f*.

[3] A detailed explanation is a required element in a free response question on the AP tests.

Points B, C, and D in the sketched graph are locations at which the concavity of f changes. You give a special name to any such point: if p is a value in the domain of a continuous function f at which f changes concavity, then you say that $(p, f(p))$ is an ***inflection point*** of f. Just as you look for locations where f changes from increasing to decreasing at points where $f'(p) = 0$ or $f'(p)$ is undefined,

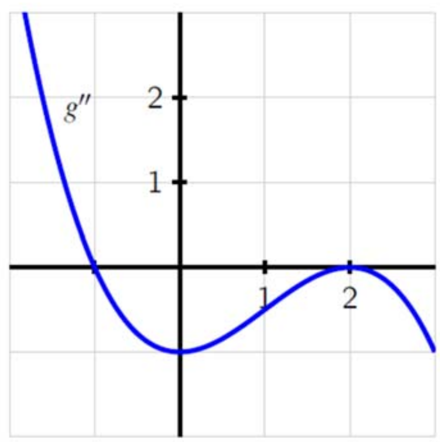


so too you find where $f''(p) = 0$ or $f''(p)$ is undefined to see if there are points of inflection at these locations.

It is important at this point in your study to remind yourself of the big picture that derivatives help to paint: the sign of the first derivative f, tells you whether the function f is increasing or decreasing, while the sign of the second derivative f'' tells you how the function f is increasing or decreasing.

Investigation 3: Suppose that g is a function whose second derivative, g'' is given by the following graph.

a) Find all points of inflection of g.



b) Fully describe the concavity of g by making an appropriate sign chart.

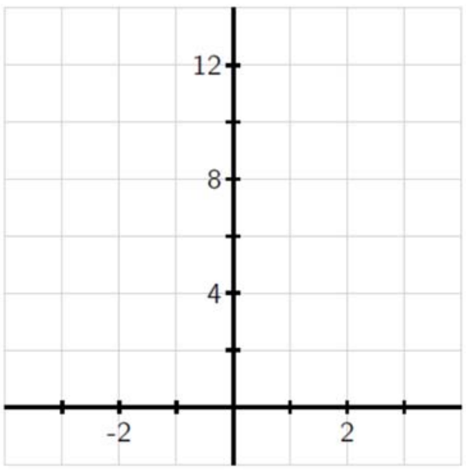
c) Suppose you are given that $g'(1.67857351) = 0$. Is there is a local maximum, local minimum, or neither (for the function g) at this critical point of g, or is it impossible to say? Why?

d) Assuming that g''(x) is a polynomial (and that all important behavior of g'' is seen in the graph above, what degree polynomial do you think $g(x)$ is? Why?

As you will see in more detail in the following section, derivatives also help you to understand families of functions that differ only by changing one or more parameters.

Investigation 4: Consider the family of functions given by $h(x) = x^2 + \cos(kx)$, where k is an arbitrary positive real number.

(a) Use a graphing utility to sketch the graph of h for several different k-values, including $k = 1, 3, 5, 10$. Plot $h(x) = x^2 + \cos(kx)$ on the axes provided below.



b) What is the smallest value of k at which you think you can see (just by looking at the graph) at least one inflection point on the graph of h?

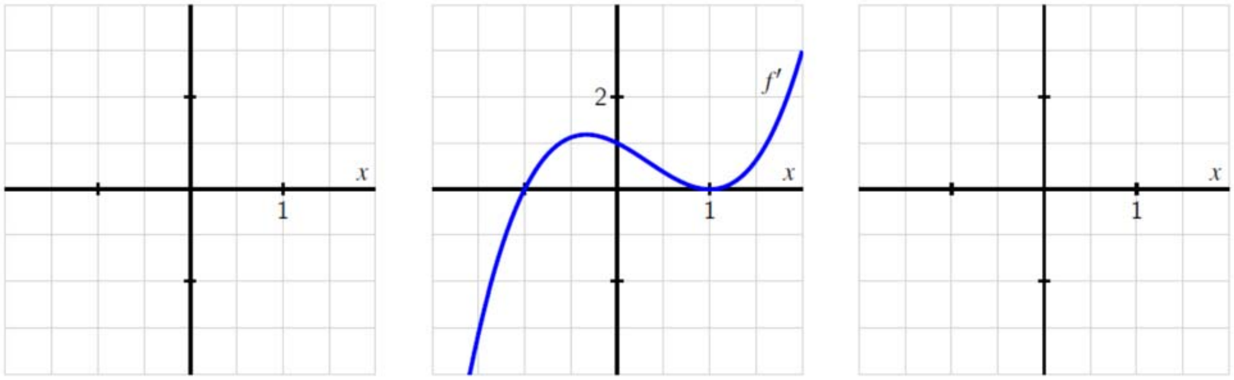
c) Explain why the graph of h has no inflection points if $k \leq \sqrt{2}$, but infinitely many inflection points if $k > \sqrt{2}$.

d) Explain why, no matter the value of k, h can only have finitely many critical numbers.

IV. Exercises

1. This problem concerns a function about which the following information is known:

- f is a differentiable function defined at every real number x
- $f(0) = -\frac{1}{2}$
- $y = f'(x)$ has its graph given at center in the figure below.



At center, a graph of y = f '(x); at left, axes for plotting y = f (x); at right, axes for plotting y = f ''(x).

a) Construct a first derivative sign chart for f. Clearly identify all critical numbers of f, where f is increasing and decreasing, and where f has local extrema.

b) On the right-hand axes, sketch an approximate graph of $y = f''(x)$.

c) Construct a second derivative sign chart for f. Clearly identify where f is concave up and concave down, as well as all inflection points.

d) On the left-hand axes, sketch a possible graph of $y = f(x)$.

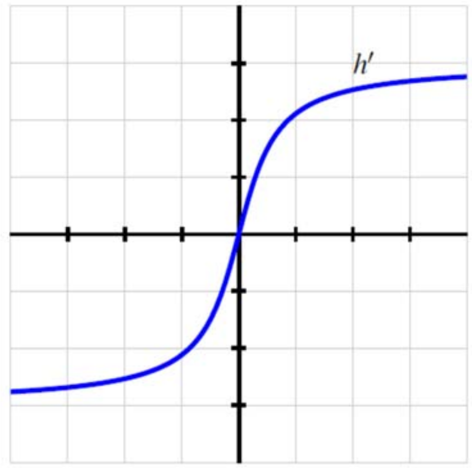
2. Suppose that g is a differentiable function and $g'(2) = 0$. In addition, suppose that on $1 < x < 2$ and $2 < x < 3$ it is known that $g'(x)$ is positive.

a) Does g have a local maximum, local minimum, or neither at $x = 2$? Why?

b) Suppose that $g''(x)$ exists for every x such that $1 < x < 3$. Reasoning graphically, describe the behavior of $g''(x)$ for x-values near 2.

c) Besides being a critical number of g, what is special about the value $x = 2$ in terms of the behavior of the graph of g?

3. Suppose that h is a differentiable function whose first derivative is given by the graph below.



a) How many real number solutions can the equation $h(x) = 0$ have? Why?

b) If $h\ (x) = 0$ has two distinct real solutions, what can you say about the signs of the two solutions? Why?

c) Assume that $\lim\limits_{x \to \infty} h'(x) = 3$, as appears to be indicated in the graph. How will the graph of $y = h(x)$ appear as $x \to \infty$? Why?

d) Describe the concavity of $y = h(x)$ as fully as you can from the provided information.

4. Let p be a function whose second derivative is $p''(x) = (x + 1)(x - 2)e^{-x}$.

a) Construct a second derivative sign chart for p and determine all inflection points of p.

b) Suppose you also know that $x = \frac{\sqrt{5}-1}{2}$ is a critical number of p. Does p have a local minimum, local maximum, or neither at $x = \frac{\sqrt{5}-1}{2}$? Why?

c) If the point $(2, \frac{12}{e^2})$ lies on the graph of $y = p(x)$ and $p'^{(2)} = -\frac{5}{e^2}$, find the equation of the tangent line to $y = p(x)$ at the point where $x = 2$. Does the tangent line lie above the curve, below the curve, or neither at this value? Why?

V. Assessments – Khan Academy

1. Complete the first ten online practice exercises in the Analyzing Functions unit

(Differentiating Common Functions) of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/analyzing-functions-with-calculus-ab?t=practice

II. Optional

1. Practice #11 challenge