2.3B To Be or Not To Be

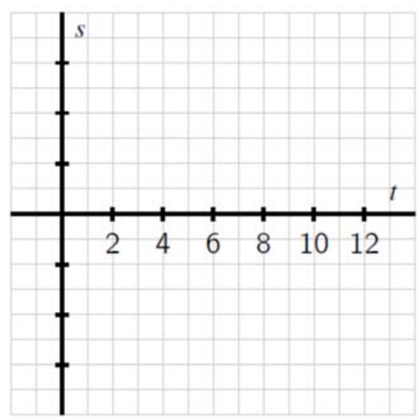
Mean Value Theorem, and the other Existence Theorems

Earlier you learned to calculate the average velocity of an object by using the formula

$$AV_{[a,b]} = \frac{s(b) - s(a)}{b - a}$$

In this lesson, you will generalize this formula for all continuous functions that are differentiable over a closed interval.

Investigation 1: A diver leaps from a 5 meter springboard. Her feet leave the board at time $t = 0$, she reaches his maximum height of 6.5 m at t = 1.1 seconds, and enters the water at $t = 2.45$. Once in the water, the diver coasts to the bottom of the pool (depth 4 m), touches bottom at $t = 7$, rests for one second, and then pushes off the bottom. From there she coasts to the surface, and takes her first breath at $t = 13$.



1a) Let $s(t)$ denote the function that gives the height of the diver's feet (in meters) above the water at time t. (Note that the "height" of the bottom of the pool is -4 meters.) Sketch a carefully labeled graph of $s(t)$ on the provided axes. Include scale and units on the vertical axis. Be as detailed as possible.

b) Based on your graph in (a), what is the average velocity of the diver over the following time intervals:

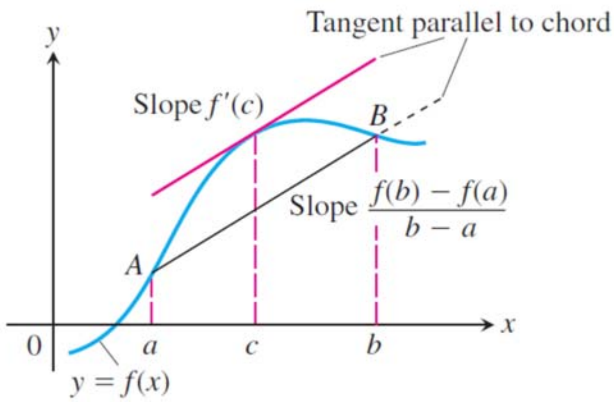
i) [2.45, 7]

ii) [0, 7]

iii. [0,13]

II. The Mean Value Theorem

The Mean Value Theorem states that somewhere between points A and B on a differentiable curve, there is at least one tangent line parallel to chord AB as illustrated below.



> **Mean Value Theorem for Derivatives:** If $f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point on the open interval (a, b), then there is at least one point c in $[a, b]$ at which
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Note that the Mean Value Theorem doesn't tell you what c is. It only tells you that there is at least one number c that will satisfy the conclusion of the theorem. To find the value of c, complete the calculations in the difference quotient.

Investigation 3: In each of the following, (i.) state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and (ii.) if it does, find each value of c in the interval (a, b) that satisfies the equation $f'(c) = \frac{f(b)-f(a)}{b-a}$

3a) $f(x) = x^2 + 2x - 1$ on $[0,1]$

b) $g(x) = x^{2/3}$ on $[0,1]$

c) $h(x) = x^{1/3}$ on $[-1,1]$

d) $r(z) = \ln(z-1)$ on $[2,4]$

e) $s(y) = sin^{-1}(y)$ on $[-1,1]$

f) $t(x) = \begin{cases} \cos(x), & \text{for } -1 \le x \le 1 \\ \sin(x), & \text{for } \quad 1 \le x \le 3 \end{cases}$ on $[0, \pi]$

III. Other Existence Theorems

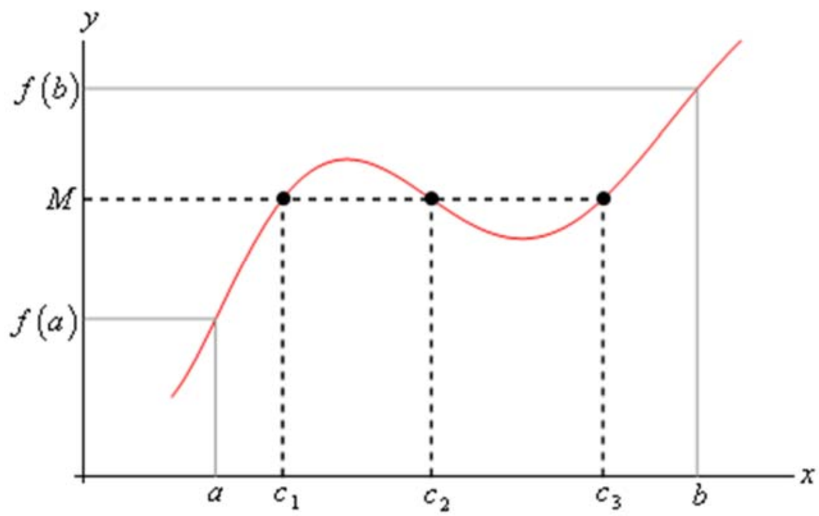
There are four existence theorems that you need to know for the AP test:

- Intermediate Value Theorem (IVT)
- Extreme Value Theorem (IVT)
- Mean Value Theorem (IVT)
- Rolle's Theorem

The IVT states that all differentiable functions must have an intermediate value property, as stated in the following theorem.

> **Intermediate Value Theorem:** If a and b are any two points in an interval on which $f(x)$ is differentiable, then $f(x)$ takes on every value between $f(a)$ and $f(b)$.

All the Intermediate Value Theorem is really saying is that a continuous function will take on all values between $f(a)$ and $f(b)$. Below is a graph of a continuous function that illustrates the Intermediate Value Theorem.



As you can see, if we pick any value, M, that is between the value of $f\,(a)$and the value of $f\,(b)$and draw a line straight out from this point the line will hit the graph in at least one point. In other words somewhere between a and b the function will take on the value of M. Also, as the figure shows the function may take on the value at more than one place.

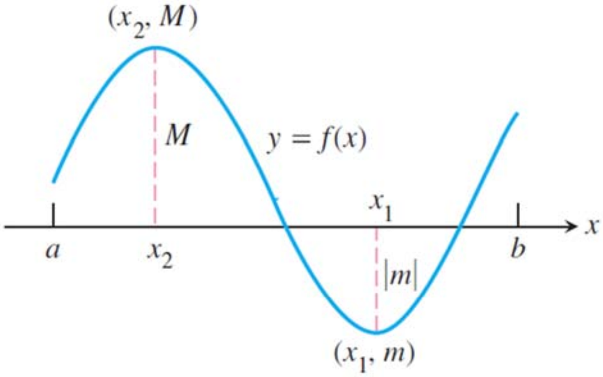
It's also important to note that the Intermediate Value Theorem only says that the function will take on the value of M somewhere between a and b. It doesn't say just what that value will be. It only says that it exists.

Finally, realize that since the derivative is a function, as well, the IVT applies to derivatives.

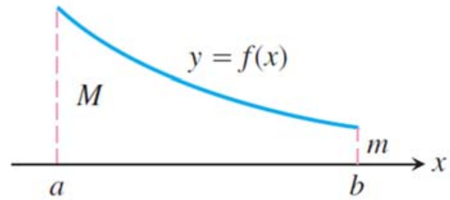
Intermediate Value Theorem for Derivatives: If a and b are any two points in an interval on which $f\,(x)$ is differentiable, then $f\,'(x)$ takes on every value between $f\,'(a)$ and $f\,'(b)$.

In the previous lesson, you examined the extreme values of a function. This can be summarized as follows:

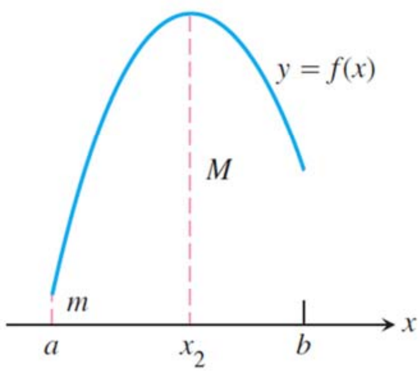
Extreme Value Theorem: If $f\,(x)$ is continuous at every point of the closed interval $[a, b]$, then $f\,(x)$ has both a maximum and minimum value on the interval $[a, b]$.



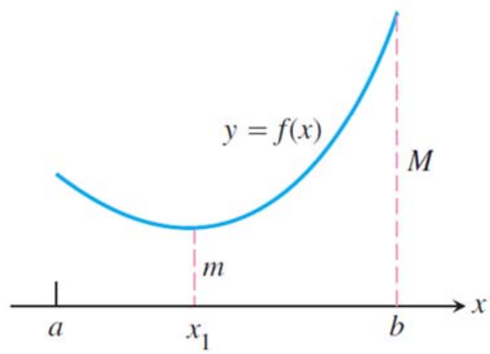
Maximum and minimum
at interior points



Maximum and minimum
at endpoints



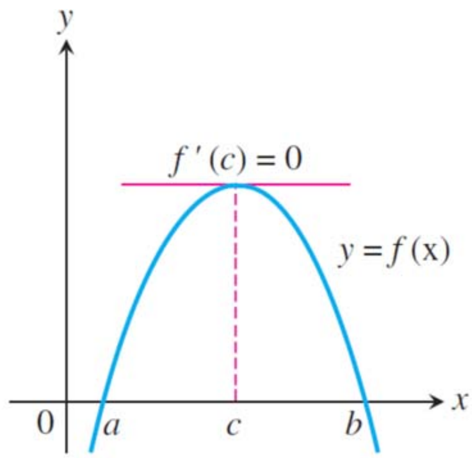
Maximum at interior point,
minimum at endpoint



Minimum at interior point,
maximum at endpoint

Again, note that the EVT applies to derivatives, as well.

The first version of the Mean Value Theorem was proved by French mathematician Michel Rolle (1652–1719). His version had $f(a) = f(b) = 0$ and was proved only for polynomials, using algebra and geometry.



Rolle's Theorem: If $f(x)$ is a function that satisfies all the following conditions:

- continuous at every point of the closed interval $[a, b]$,
- differentiable at every point (a, b), and
- $f(a) = f(b)$

then there is a c such that $a < c < b$ and $[a, b]$ and $f'(c) = 0$

In other words, $f(x)$ has a critical point (a, b), if it meets all three conditions stated above.

Example 1. Given $f(x) = x^3 - 4x$. Find a value in the interval (-2, 2) which satisfies Rolle's Theorem.

Solution. First verify that Rolle's Theorem can be used. $f(-2) = 0$ and $f(2) = 0$. Since $f(x)$ is a polynomial it is differentiable, therefore it is continuous.

Then $f'^{(x)} = 3x^2 - 4$ and there must be at least one value of x between -2 and 2 where $f'^{(x)} = 0$.

This is found by solving the equation, $0 = 3x^2 - 4$ and getting $x^2 = \frac{4}{3}$. Therefore there are two critical values at $x = \pm\frac{2\sqrt{3}}{3}$.

V. Exercises

1.In each of the following, (i.) state whether or not the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and (ii.) if it does, find all values of c in the interval (a, b) that satisfies the equation $f'(c) = \frac{f(b)-f(a)}{b-a}$

a) $f(x) = x^2$ on $[-2,1]$

b) $g(x) = \sqrt{x-2}$ on $[2,6]$

c) $h(x) = |x-1|$ on $[0.4]$

d) $w(x) = \begin{cases} \sin^{-1}(x), & \text{for } -1 \le x \le 1 \\ \frac{x}{2} + 1(x), & \text{for } 1 \le x \le 3 \end{cases}$ on $[-1,3]$

2. Explain why the Mean Value Theorem does not apply to the function $f(x) = \dfrac{1}{x-3}$ on the interval $[0,6]$

3. Show that the function $f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 \le x \le 1 \end{cases}$ is not the derivative of any function on the interval $-1 \le x \le 1$.

4. The height of an object t seconds after it is dropped from a height of 500 meters is $s(t) = -4.9t^2 + 500$.

a) Find the average velocity of the object during the first 3 seconds (in other words during the interval $[0,3]$).

b) Use the Mean Value Theorem to verify that at some time during the first 3 seconds of fall, the instantaneous velocity equals the average velocity. Find the time.

Sample AP Test Questions:

5. The function $f(x) = x^{\frac{2}{3}}$ on $[-8,8]$ does not satisfy the conditions of the Mean Value Theorem because

A) $f(0)$ is not defined
B) $f(x)$ is not continuous on $[-8,8]$
C) $f'(-1)$ does not exist
D) $f(x)$ is not defined for $x < 0$
E) $f'(0)$ does not exist

6. At how many points on the interval $[0,\pi]$ does $f(x) = 2\sin x + \sin 4x$ satisfy the Mean Value Theorem?

A) none B) 1 C) 2 D) 3 E) 4

7. A balloon is being filled with helium at the rate of 4 ft^3/min. The rate, in square feet per minute, at which the surface area is increasing when the volume is $\frac{32\pi}{3}$ ft^3 is

A) 4π B) 2 C) 4 D) 1 E) 2π

8. Which statement below is true about the curve $y = \frac{2x^2+4}{2+7x-4x^2}$?

A) The line $x = -\frac{1}{4}$ is a vertical asymptote

B) The line $x = 1$ is a vertical asymptote

C) The line $y = -\frac{1}{4}$ is a horizontal asymptote

D) The graph has no vertical or horizontal asymptote

E) The line $x = 2$ is a horizontal asymptote