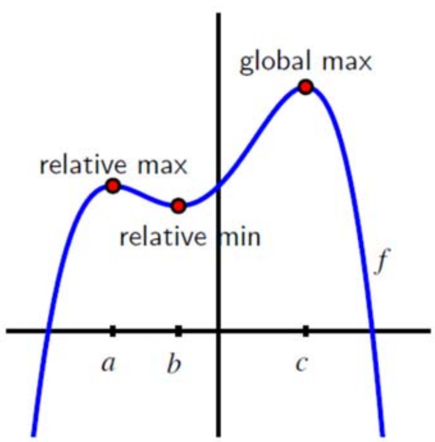
2.34 Max Effort with the EVT

Global Optimization

You have seen that you can use the first derivative of a function to determine where the function is increasing or decreasing, and the second derivative to know where the function is concave up or concave down. Each of these approaches provides you with key information that helps you determine the overall shape and behavior of the graph, as well as whether the function has a relative minimum or relative maximum at a given critical number.

Remember that the difference between a relative maximum and a global maximum is that there is a relative maximum of f at $x = p$ if $f(p) \geq f(x)$ for all x near p, while there is a global maximum at p if $f(p) \geq f(x)$ for all x in the domain of f.



For instance, at left you see a function f that has a global maximum at $x = c$ and a relative maximum at $x = a$, since $f(c)$ is greater than $f(x)$ for every value of x, while f a is only greater than the value of $f(x)$ for x near a. Since the function appears to decrease without bound, f has no global minimum, though clearly f has a relative minimum at $x = b$.

Your emphasis in this section is on finding the global extreme values of a function (if they exist). In so doing, you will either be interested in the behavior of the function over its entire domain or on some restricted portion. The former situation is familiar and similar to work that you did in the two preceding sections of the text.

Investigation 1: Let $f(x) = 2 + \frac{3}{1+(x+1)^2}$.

a) Determine all of the critical numbers of f.

b) Construct a first derivative sign chart for f and thus determine all intervals on which f is increasing or decreasing.

c) Does f have a global maximum? If so, why, and what is its value and where is the maximum attained? If not, explain why.

d) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

e) Explain why $f(x) > 2$ for every value of x.

f) Does f have a global minimum? If so, why, and what is its value and where is the minimum attained? If not, explain why.

II. Global Optimization

For the functions in Investigation 1, you were interested in finding the global minimum and global maximum on the entire domain, which turned out to be $(-\infty, \infty)$for each. At other times, your perspective on a function might be more focused due to some restriction on its domain. For example, rather than considering $f(x) = 2 + \frac{3}{1+(x+1)^2}$ for every value of x, perhaps instead you are only interested in those x for which $0 \leq x \leq 4$ and you would like to know which values of x in the interval $[0, 4]$ produce the largest possible and smallest possible values of f. You are accustomed to critical numbers playing a key role in determining the location of extreme values of a function; now, by restricting the domain to an interval, it makes sense that the endpoints of the interval will also be important to consider, as you see in the following activity. When limiting yourselves to a particular interval, you will often refer to the absolute maximum or minimum value, rather than the global maximum or minimum.

Investigation 2: Let $g(x) = \frac{1}{3}x^3 - 2x + 2$.

(a) Find all critical numbers of g that lie in the interval $-2 \leq x \leq 3$.

(b) Use a graphing utility to construct the graph of g on the interval $-2 \leq x \leq 3$.

(c) From the graph, determine the *x*-values at which the absolute minimum and absolute maximum of g occur on the interval [-2, 3].

(d) How do your answers change if you instead consider the interval $-2 \leq x \leq 2$?

(e) What if you instead consider the interval $-2 \leq x \leq 1$?

In Investigation 2, you saw how the absolute maximum and absolute minimum of a function on a closed, bounded interval $[a, b]$, depend not only on the critical numbers of the function, but also on the selected values of a and b. These observations demonstrate several important facts that hold much more generally. First, you state an important result called the Extreme Value Theorem.

The Extreme Value Theorem (EVT): If *f* is a continuous function on a closed interval $[a, b]$, then *f* attains both an absolute minimum and absolute maximum on $[a, b]$. That is, for some value x_m such that $a \leq x_m \leq b$, it follows that $f(x_m) \leq f(x)$ for all x in $[a, b]$. Similarly, there is a value x_M such that in [a, b], such that $f(x_M) \geq f(x)$ for all x in $[a, b]$. Letting $m = f(x_m)$ and $M = f(x_M)$; it follows that $m \leq f(x) \leq M$ for all x in $[a, b]$

The Extreme Value Theorem tells you that provided a function is continuous, on any closed interval $[a, b]$ the function has to achieve both an absolute minimum and an absolute maximum. Note, however, that this result does not tell you where these extreme values occur, but rather only that they must exist. As seen in the examples of Investigation 2, it is apparent that the only possible locations for relative extremes are either the endpoints of the interval or at a critical number (the latter being where a relative minimum or maximum could occur, which is a potential location for an absolute extreme). Thus, you have the following approach to finding the absolute maximum and minimum of a continuous function f on the interval $[a, b]$:

- find all critical numbers of f that lie in the interval;
- evaluate the function *f* at each critical number in the interval and at each endpoint of the interval;
- from among the noted function values, the smallest is the absolute minimum of *f*
- on the interval, while the largest is the absolute maximum.

Investigation 3: Find the *exact* absolute maximum and minimum of each function on the stated interval.

a) $h(x) = xe^{-x}$, $[0, 3]$

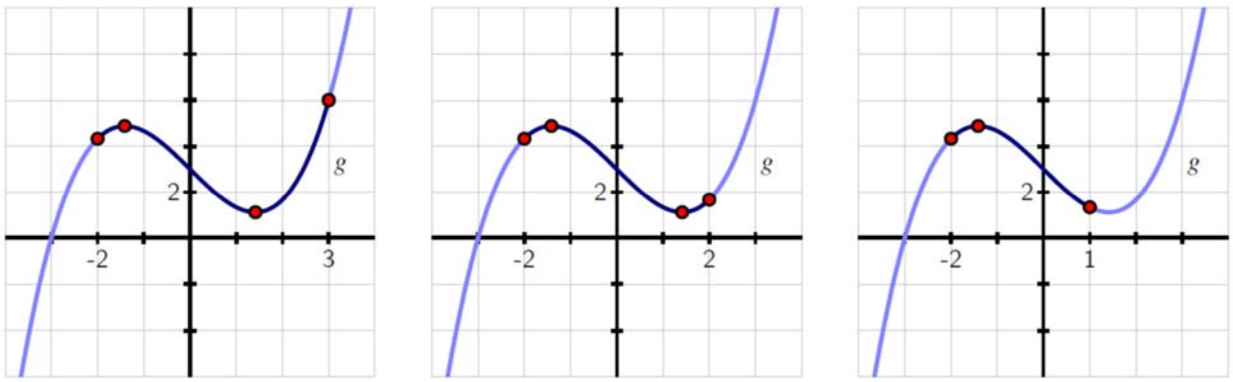
b) $p(t) = \sin(t) + \cos(t)$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$

c) $q(x) = \frac{x^2}{x-2}$, $[3,7]$

(d) $f(x) = 4 - e^{-(x-2)^2}$, $[-\infty, \infty]$

(e) $h(x) = xe^{-3x}$, $\left[0, \frac{2}{3}\right]$

One of the big lessons in finding absolute extreme values is the realization that the interval you choose has nearly the same impact on the problem as the function under consideration. Consider, for instance, the function pictured below.



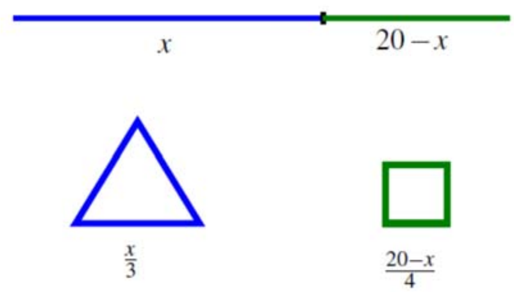
In sequence, from left to right, as you see the interval under consideration change from [-2, 3] to [-2, 2] to [-2, 1], you move from having two critical numbers in the interval with the absolute minimum at one critical number and the absolute maximum at the right endpoint, to still having both critical numbers in the interval but then with the absolute minimum and maximum at the two critical numbers, to finally having just one critical number in the interval with the absolute maximum at one critical number and the absolute minimum at one endpoint. It is particularly essential to always remember to only consider the critical numbers that lie within the interval.

III. Moving towards applications

In the next lesson, you will focus almost exclusively on applied optimization problems: problems where you seek to find the absolute maximum or minimum value of a function that represents some physical situation. You will conclude this preliminary lesson with an example of one such problem because it highlights the role that a closed, bounded domain can play in finding absolute extrema. In addition, these problems often involve considerable preliminary work to develop the function which is to be optimized, and this example demonstrates that process.

Example 1. A 20 cm piece of wire is cut into two pieces. One piece is used to form a square and the other an equilateral triangle. How should the wire be cut to maximize the total area enclosed by the square and triangle? to minimize the area?

Solution. You begin by constructing a picture that exemplifies the given situation. The primary variable in the problem is where you decide to cut the wire. You thus label that point *x*, and note that the remaining portion of the wire then has length 20 - *x* As shown in the fat right, you see that the x cm of the wire that are used to form the equilateral triangle result in a triangle with three sides of length $\frac{x}{3}$ cm . For the remaining 20 - *x* cm of wire, the square that results will have each side of length $\frac{20-x}{4}$.



At this point, note that there are obvious restrictions on x: in particular, $0 \leq x \leq 20$. In the extreme cases, all of the wire is being used to make just one figure. For instance, if x = 0, then all 20 cm of wire are used to make a square that is 5 × 5.

Now, the overall goal is to find the absolute minimum and absolute maximum areas that can be enclosed. Notice that the area of the triangle is

$$A_{\Delta} = \frac{1}{2}bh = \frac{1}{2} \cdot \frac{x}{3} \cdot \frac{x\sqrt{3}}{6},$$

since the height of an equilateral triangle is $\sqrt{3}$ times half the length of $\frac{x}{3}$, the base. Further, the area of the square is

$$A_{\blacksquare} = s^2 = \left(\frac{20-x}{4}\right)^2$$

Therefore, the total area function is

$$A(x) = \frac{\sqrt{3}x^2}{36} + \left(\frac{20-x}{4}\right)^2$$

Again, note that you are only considering this function on the restricted domain [0, 20] and you seek its absolute minimum and absolute maximum.

Differentiating $A(x)$, you have

$$A'(x) = \frac{\sqrt{3}x}{18} + 2\left(\frac{20-x}{4}\right)\left(\frac{1}{4}\right)$$

$$= \frac{\sqrt{3}}{18}x + \frac{1}{8}x - \frac{5}{2}$$

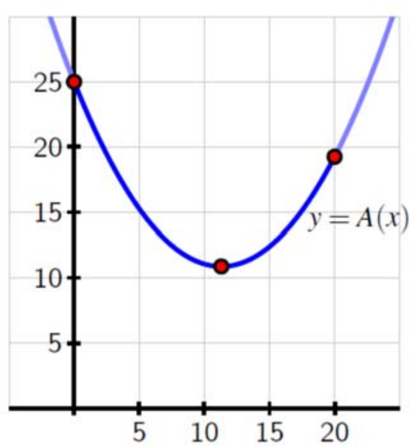
Setting $A'(x) = 0$, it follows that $x = \frac{180}{4\sqrt{3}+9} \approx 11.301$ is the only critical number of A, and you also note that this lies within the interval [0, 20].

Evaluating A at the critical number and endpoints, you see that

- $A\left(\frac{180}{4\sqrt{3}+9}\right) = \frac{\sqrt{3}\left(\frac{180}{4\sqrt{3}+9}\right)^2}{4} + \left(\frac{20-\frac{180}{4\sqrt{3}+9}}{4}\right)^2 \approx 10.874$
- $A(0) = 25$
- $A(20) = \frac{\sqrt{3}}{36}(400) = \frac{100}{9}\sqrt{3} \approx 19.245$

Thus, the absolute minimum occurs when $x \approx 11.301$ and results in the minimum area of approximately 10.874 square centimeters, while the absolute maximum occurs when you invest all of the wire in the square (and none in the triangle), resulting in 25 square centimeters of area.

These results are confirmed by a plot of $y = A(x)$ on the interval [0, 20], as shown below.



Investigation 2: A piece of cardboard that is 10 x 15 (each measured in inches) is being made into a box without a top. To do so, squares are cut from each corner of the box and the remaining sides are folded up. If the box needs to be at least 1 inch deep and no more than 3 inches deep, what is the maximum possible volume of the box? what is the minimum volume? Justify your answers using calculus.

a) Draw a labeled diagram that shows the given information. What variable should you introduce to represent the choice you make in creating the box? Label the diagram appropriately with the variable, and write a sentence to state what the variable represents.

b) Determine a formula for the function V (that depends on the variable in (a)) that tells you the volume of the box.

c) What is the domain of the function V? That is, what values of x make sense for input? Are there additional restrictions provided in the problem?

d) Determine all critical numbers of the function V.

e) Evaluate V at each of the endpoints of the domain and at any critical numbers that lie in the domain.

f) What is the maximum possible volume of the box? the minimum?

IV. Exercises

1. Based on the given information about each function, decide whether the function has global maximum, a global minimum, neither, both, or that it is not possible to say without more information. Assume that each function is twice differentiable and defined for all real numbers, unless noted otherwise. In each case, write one

sentence to explain your conclusion.

a) f is a function such that $f''(x) < 0$ for every x.

b) g is a function with two critical numbers a and b (where $a < b$), and $g'(x) < 0$ for $x < a$, $g'(x) < 0$ for $a < x < b$, and $g'(x) > 0$ for $x > b$.

c) h is a function with two critical numbers a and b (where $a < b$), and $h'(x) < 0$ for $x < a$, $h'(x) > 0$ for $a < x <$ b, and $h'(x) < 0$ for $x > b$.
In addition, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

(d) p is a function differentiable everywhere except at $x = a$ and $p''(x) > 0$ for $x < a$ and $p''(x) < 0$ for $x > a$.

2. For each of the functions described below (each continuous on $[a, b]$), state the location of the function's absolute maximum and absolute minimum on the interval $[a, b]$, or say there is not enough information provided to make a conclusion. Assume that any critical numbers mentioned in the problem statement represent all of the critical numbers the function has in $[a, b]$. In each case, write one sentence to explain your answer.

a) $f'(x) \leq 0$ for all x in $[a, b]$

b) g has a critical number at c such that $a < c < b$ and $g'(x) > 0$ for $x < c$ and

$g'(x) < 0$ for $x > c$

c) $h(a) = h(b)$ and $h''(x) < 0$ for all x in $[a, b]$

d) $p(a) > 0$, $p(b) < 0$, and for the critical number c such that $a < c < b$,
$p'(x) < 0$ for $x < c$ and $p'(x) > 0$ for $x > c$

3. Let $s(t) = 3\sin\left(2\left(t - \frac{\pi}{6}\right)\right) + 5$. Find the exact absolute maximum and minimum of s on the provided intervals by testing the endpoints and finding and evaluating all relevant critical numbers of s.

(a) $[\frac{\pi}{6}, \frac{7\pi}{6}]$

(b) $[0, \frac{\pi}{2}]$

(c) $[0, 2\pi]$

(d) $[\frac{\pi}{3}, \frac{5\pi}{6}]$