Georg Cantor: Champion of the Infinite

This view [of the infinite], which I consider to be the sole correct one, is held by only a few. While possibly I am the very first in history to take this position so explicitly, with all of its logical consequences, I know for sure that I shall not be the last!

Georg Cantor (German mathematician; 1845 - 1918)

My theory stands as rm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all because I have followed it roots, so to speak, to the first infallible cause of all created things.

Georg Cantor (German mathematician; 1845 - 1918)

Georg Cantor was the eldest of six children born to Georg Woldemar Cantor and Maria Anna Bohm. He was raised as a strict Lutheran in St. Petersburg, Russia, then Wiesbaden and Frankfurt, Germany. His father was a successful businessman despite several difficult failures. In addition to instilling strong religious faith and diligent work ethic in his children, the elder Georg Cantor also expected his children to be active students of the liberal arts and sciences. This included the study of music, an area where many relatives were well known, literature, art, mathematics, the physical sciences and the natural sciences. In the elder Cantor's own words:

To the procurement of diverse, thorough, scientific and practical knowledge; to the perfect acquisition of foreign languages and literatures; to the many-sided development of the mind in many humanistic disciplines { and of this you must always be thoroughly conscious! { to all this the second period of your life, your youth, now just beginning,

is destined, in order rst to equip yourself with dignity by means of all this for those struggles yet to come.

While a strong supporter of his children, he had unusually high expectations for his firstborn son and namesake Georg. In a letter to the then fifteen year old Georg on the occasion of his confirmation (the same letter quoted above) the elder Cantor seemed to predict the course of his son's future:

> How often the most promising individuals are defeated after a tenuous, weak resistance in their first serious struggle following their entry into practical affairs. Their courage broken, they atrophy completely thereafter, and even in the best case they will still be nothing more than a so-called ruined genius! ...But they lacked that steady heart, upon which everything depends!... This sure heart, which must live in us, is: a truly religious spirit!

> ...I close with these words: Your father, or rather your parents and all other members of the family both in Germany and in Russia and in Denmark have their eyes on you as the eldest, and expect you to be nothing less than a Theodor Schaeffer [young Cantor's teacher] and, God willing, later perhaps a shining star on the horizon of science.

By 1862 Cantor's studies had come to a crossroads. He was drawn to mathematics, but was worried that this calling was not among his father's expectations. His father, who would die a short time later in his mid-fifties, was in fact supportive of Cantor's desire to pursue mathematics. Georg's glowing reaction to this support, a state that would later become part of a cycle of manic highs and lows, is evident in a letter to his father:

> My Dear Papa! You cannot imagine how very happy your letter made me; it determines my future. The last few days have left me in doubt and uncertainty. I could reach no decision. My sense of duty and my own wishes fought continuously one against the other. Now I am happy when I see that it will not longer distress you if I follow my own feelings in this decision. I hope that you will still be proud of me one day, dear Father, for my soul, my entire being lives in my calling; whatever one wants and is able to do, whatever it is toward which an unknown, secret voice calls him, that he will carry through to success.

For the next several years Cantor studied at the University of Berlin under some of the greatest living mathematicians, including: Kummer, Kronecker, and Weierstrass. After passing the required exams he became a Privatdozent at the university in Halle, near Leipzig. Originally interested in number theory, Cantor soon turned toward analysis where critical progress was being made in understanding the foundations of calculus after a 200 year struggle that had seen little or no previous progress.

In Cantor's work the nature and structure of the real number system and its subsets proved to be a critical and insufficiently understood landscape. By 1872 he had essentially solved one of the main open problems in the area of trigonometric series. In solving this problem Cantor realized that "the number concept...carries within it the germ of a necessary and absolutely infinite extension." His investigation of the infinite, or the transfinite as Cantor called it, that this solution necessitated would occupy the majority of Cantor's subsequent mathematical life.

Cantor's work on the infinite was as revolutionary as it was unexpected and controversial. Cantor was almost single-handedly responsible not only for the germination of the critical ideas that brought rigor and certainty to the study of the infinite, but also for unceasing devotion to nurturing this theory so that "once mathematicians were ready to consider the significance of the transfinite numbers, the entire theory would be ready to stand on the foundations he had given it." So complete and revolutionary was this development that it is unparalleled in scope and significance by any event in the history of mathematics short of Newton's invention of calculus. Yet this revolution was not completed without considerable hardship on Cantor. The infinite and the infinitesimal had been banished by many mathematicians as the root cause of the unstable

foundation for calculus. Cantor's work met vigilant resistance at every stage. The strongest resistance came from the German mathematical community lead by Leopold Kronecker (German mathematician;1823 - 1891). Cantor was unable to lure prominent mathematicians to collaborate with him at the Halle and was unable to secure a prominent position at a more prestigious German university even as his monumental efforts were being recognized by mathematicians and many learned societies outside of Germany.

In addition Cantor's strong philosophical and theological beliefs made his ongoing defense of transfinites that much more difficult. His transfinites were broadly regarded as heretical – contrary to the existence of the Almighty. For such a deeply faithful man raised in the strict Lutheran tradition this was unacceptable. Hence, his efforts to secure the proper place of the transfinites took on a critical theological perspectives as well, and he addressed this issue in detail in many publications. Cantor's theory met with philosophical objections which he felt compelled to address. Finally, perhaps due in part to his father's hope that Cantor would devote himself to a more applicable calling than pure mathematics, Cantor attempted to use his transfinites to develop an "organic explanation of nature." Cantor was compelled to defend his transfinites against the attacks of mathematicians, theologians, philosophers, and scientists simultaneously.

Cantor's great strength and determination, religious faith, and profound intellectual calling – instilled in him since childhood – allowed him to make remarkable progress in uncovering the mystery of the infinite and defending it against myriad attacks from many fronts for some thirty years. Unfortunately, a fragile mental state did not allow Cantor to continue to champion his theory of transfinites effectively past the age of sixty. Recent evidence suggests Cantor's mental difficulties can be attributed to manic depression. His first serious mental breakdown was in 1884. While he did not suffer another serious breakdown until 1899, the same year his son died four days before his fourteenth birthday, his anxiety about opposition to his theory and perceived persecution increased dramatically. By the time of the Third International Congress for Mathematicians in 1904 he had been in the sanitarium twice more for extended periods. At this Congress Jules Konig presented a paper that appeared to destroy the heart of Cantor's theory. Cantor fumed, hounding notable mathematicians to regale them with possible stumbling blocks in Konig's theory. While Konig's argument eventually was shown to be poorly grounded, this episode marked a final turning point in Cantor's defense of the transfinites. After this he spent more and more time in sanitariums, often for periods close to a year. Shortly before his

death on 6 January, 1918, at the age of 73, Cantor wrote a poem to his wife which ends as follows:

> To suffer gladly, pen a poem,
> To escape the world I'm in.

Perhaps the most sticking irony in the celebrated yet troubled life of Georg Cantor is the nature of two mathematical difficulties that plagued his theory so strongly but resisted his greatest efforts. Cantor's theory of transfinites relied heavily on the extant theory of sets. By the turn of the century there were many paradoxes that Cantor's theory seemed to give rise to. It is now quite clear that it was not Cantor's theories of the infinite that gave rise to these paradoxes, rather it is our naive notions of set theory. These notions and the extant theory contain in their most basic elements a host of damaging paradoxes that arise even in finite situations. Second, the key missing result in Cantor's entire theory was what we referred to as the Continuum Hypothesis at the end of the previous chapter. His inability to obtain closure on this key result was certainly a devastating failure to him. But, as we have said, herein lies the ultimate irony - within the confines of typical set theory, the Continuum Hypothesis is undecidable!