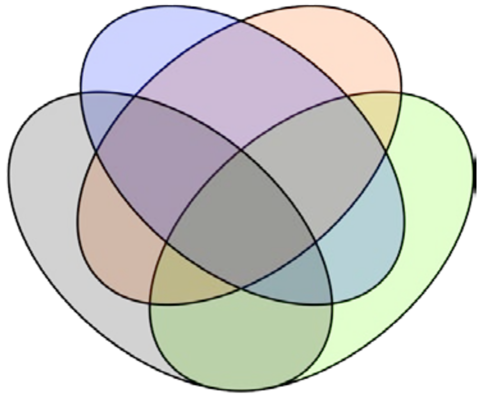

Set Concepts

Terminology of Sets

A set is a well-defined, unordered collection of objects having no duplicate members. When we use the term **well-defined set**, we mean that membership in the collection is unambiguous, not open to interpretation, and strictly capa-ble of determination through an investigation of the facts. This concept, when viewed with the greatest possible generality, is a truly fundamental notion for us, since all of mathematics is constructed on it.

As an illustration, "the set of all legal residents of your town" is a well- defined set because legal stipulations dictate who is and who is not a legal resident of your town. Confronted by any particular individual you meet, direct examination of the facts will enable you to determine whether that person is a legal resident of your town and hence considered to be an object within the set. Alternatively, "the set of all smart people in your town" is not well defined because the perception of what it means to be a smart person is open to debate, and may not be interpreted the same way by different people. Consequently, whether John Smith is a legal resident of your town can be definitively established, while whether John Smith is a smart person in your town cannot.

Your Turn: Explain why each of the following sets are, or are not, well defined:

1. The set of tall people in your class

2. The set of letters in the English alphabet

3. The set of warm days in the past year

4. The set of libraries in New Jersey having more than 1000 library card holders.

Set Notation

Let *a* be an object in set *A*. The notation $a \in A$ is used to indicate that the object a is a **member** of set A or that a is an **element** of A. The notation $a \notin A$. A is used to say that the object a is not a member of set A. It is conceivable, of course, that a set could have no members, and the symbol Ø is used to denote such a set, which we refer to as the empty set or the **null set**.

Membership in a set can be specified in at least three ways. The first way is through a **verbal description**, wherein the members of the set are described in an unambiguous manner. An illustration would be "*M* is the set of all individu-als currently on the roster of the UConn Women's Basketball team:" This set clearly is well defined. If I ask, for instance, whether Michael Jordan is a member of set *M*, the answer can be definitively determined through investiga-tion of the facts.

The second way of expressing the membership of a set is called **roster notation**, wherein the members of the set are listed for us. Roster form is partic-ularly useful if the membership of a set is fairly limited in size or if the members follow an easily recognizable pattern. An example of roster notation would be *A* = { 1, 2, 3, 4}. Should the list of members be rather substantial, we can use an ellipsis to shorten the roster, provided that we can present the roster in such a way that membership in the set is clear. An ellipsis is a succession of three "dots" that indicate that the demonstrated pattern of numbers continues, either for-ever or until a number following the ellipsis is reached. For instance, we could say *B* = {2, 4, 6, ..., 100} if we wanted to indicate the even natural numbers less than or equal to 100, but it would be improper to attempt to give the same set as *B* = {2, ..., 100}, since we are not provided with enough information to be certain of the set's membership criteria.

The third way of describing set membership is called **set-builder notation**, which looks like this: *P* = {*x* I *x* is an even number}. The information within the braces tells us that "generic" members of the set *P* will be referred to as *x* and that the form of the elements *x* is described by the rule following the vertical line. We refer to that as a characteristic property or characteristic trait of the members of the set. In this case, the members of set *P* are those numbers that happen to be multiples of 2.

The set-builder notation form is read in a particular way. For example, we would read the preceding illustration, *P* = {*x* I *x* is an even number}, as "*P* is equal to the set of all ele-ments *x* such that *x* is an even number:' This takes some getting used to, and we encourage you to think carefully about how set-builder notation would be read whenever you encounter it. That practice will provide you with the familiarity you need to make this potentially strange and new notation less intimidating.

Your Turn: Give a verbal description of the following examples of sets given in set-builder notation:

5. $A = \{x \in N | 5 < x < 6\}$;

6. $B = \{x | x$ is any of points common to any two distinct parallel lines in a plane$\}$.

It may not be immediately apparent, but the two sets we have just presented are empty. That they are empty becomes evident on examination of the criteria for membership. For the set A, it is impossible for a natural number to occur between 5 and 6, and thus there are no elements capable of satisfying the defini-tion for membership in A. That B is empty is a consequence of the definition of parallel lines in Euclidean geometry, where a postulate dictates that distinct parallel lines can share no points. Consequently, we could say $A = B = \varnothing$ .Note that the two sets are not obviously equal when first presented but are found to be so only on further examination.

Within mathematics, you should become familiar with particular sets of numbers, as they form a rich source of examples. Each of these sets is represented by a particular boldfaced capital letter, and each of those notations should be reserved for its particular set.

Some of the Sets of Numbers

N = the set of natural numbers = { 1, 2, 3, ...}
W = the set of whole numbers = {O, l, 2, 3, ...}
Z = the set of integers = {..., -3, -2, -1, 0, l, 2, 3, ...}

There are other sets of numbers: the rational numbers (symbolized by *Q*), the irrational numbers (for which there is no universally accepted set name), the real numbers (symbolized by *R*), and the complex numbers (sym-bolized by *C*).

7. For the set G = $\{x \in W \mid x \leq 7\}$, describe the members of the set using roster form and also a verbal description.

Universal Sets

It is customary to define an all-encompassing set called the **universal set**, symbolize by U, from which all possible sets must take their members. An exam-ple of a universal set could be a particular set of numbers or a particular group of individuals. If no universal set is specified, it may be that its composition is irrelevant for the purposes of the problem, so we can take it to be some larger set within which the members of our set reside.

Once the universal set has been specified, we will consider only members from that set for the duration of the problem under consideration. If, for example, we define $U = \{l, 2, 3, 4\}$ then for the remainder of the problem, no other objects exist. Thus, using that universal set, if we then wanted to create a set consisting of all even numbers from the universal set and call that set *E*, we would say $E = \{2, 4\}$. Although there are other even numbers from the broader context of all possible numbers, from the perspective of our universal set, these are the only even numbers in existence.

Cardinality of a Set

When examining sets, we will want to consider various features, including the **cardinality** of the set. Cardinality refers to the quantity of members belong-ing to the set, and it is represented by a symbol that is similar to the familiar notation of absolute value: $|S|$. An alternative notation used in some texts is the symbol *n(S)*, which also represents the cardinality of set *S*.

Cardinality may be **finite** (that is, membership in the set is limited to a particular, possibly large quantity of members) or **infinite** (there are an unlimited number of members in the set). The cardinality of the empty set is zero, since that set has been defined to have no members.

Your Turn: Determine if the following sets are finite or infinite:

8. The set of players currently on the roster of the New York Mets baseball team

9. The set of grains of sand existing on all the beaches of the planet Earth

10. The set of blood cells in your body at any given time

11. The set of numbers between 0 and 2.

Equality and Equivalence of Sets

When we consider two separate sets, we can establish the relationships of equality and equivalence between them. Two sets *A* and *B* are said to be **equal**, symbol-ized $A = B$, if their membership is entirely identical (keeping in mind that the order of the elements in the sets need not be the same). Two sets *A* and *B* are said to be **equivalent**, symbolized $A \neq B$, if they share the same cardinality. Note that if two sets are equal, they are also equiva-lent, but the converse is not true. For instance, $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$ are equivalent sets, since both have cardinality 4, but they are not equal, since they do not have the same members.

Your Turn: Are the following sets equal, equivalent, both, or neither?

12. The set of all letters in the English alphabet and the set of whole numbers between O and 25, inclusive

13. The sets A = {1,2, 3, 4} and B = {3, 4, 2, 1}

14. The sets C = {3, 1, 4, 5} and D = {a, b, c, d, e}

The Complement of a Set

Once the universal set has been established, possibly by inference or implica-tion, and a set A described, the universal set is subdivided naturally into two groups: those members of the universal set lying within A and those not lying within A. The members of the universal set that do not fall within the set A are said to form the **complement** of A, denoted A^c or A'.
Let's suppose U = {1, 2, 3, ..., 10} and A = {1, 2, 3, 4}. Then A' = {5, 6, 7, 8, 9, 10}. Note that the set U is completely subdivided by a set and its complement in the sense that all members of U must lie in one of those two sets.

15. Let U = the set of all whole numbers between 10 and 20 and A = {11, 13, 14, 17}. What is A'?

16. Let U = the set of all letters in the English alphabet, and A = {a, e, i, o, u}. What is A' ?

Practice

In questions 1-7, answer with complete sentences and correct grammar and spelling.

1. What is a set?

2. What does it mean to say two sets are equal?

3. What does it mean to say two sets are equivalent?

4. What is meant by the cardinality of a set?

5. What is the empty set, and how is it symbolized?

6. What does it mean to say a set is finite?

7. What is an ellipsis, and what does it represent?

In questions 8-12, determine if the sets are well defined. If they are not well defined, state why.

8. The set of paid employees of the U.S. government.

9. The set of most efficient computer brands.

10. The set of well-spoken professors at Harvard University.

11. The set of astronauts who have piloted the space shuttle *Atlantis*.

12. The set of even integers between 6 and 7.

In the following problems, determine if the sets are finite or infinite. If they are finite, state their cardinality.

13. The set of states in the United States of America at the present time

14. The set of digits in the number "1 trillion"

15. The set of numbers between 4 and 10

Express the following sets in roster form and state the cardinality of the set.

16. The set of letters in the word "Mississippi"

17. The set of all natural numbers less than 50

18. The set of all states in the United States whose names begin with the letter N

19. $A = \{x \mid 2 - x = 7\}$

20. The set of all living persons in the United States who hold or have held the office of president of the United States

Express the following sets in set-builder notation.

21. *A* = { l, 2, 3}

22. *B* = {O, 2, 4, ...}

23. C = {2, 3, 5, 7, 11, 13}

24. D is the set of all months in the year having exactly 20 days.

25. E is the set of all odd natural numbers less than 1000.

Give a verbal description of the members of the following sets

26. {3, 6, 9, 12, 15}

27. {Fred, Barney, Betty, Wilma}

28. $\{ x \in W \mid 2 < x \leq 6 \}$

29. {Hawaii, Alaska}

30. {l, 3, 5, 7, ..., 19}

For the following problems, use the sets *A* = {2, 4, 6, 8, 10}, *B* = {3, 4, 5, 6}, and *C* = {a, b, c, d}.

31. Determine I*A*I

32. Determine I*B*I

33. Determine I*C*|

For the following problems, determine if the given sets are equal, equivalent, both, or neither.

34. *B* = {3, 4, 5, 6}, *C* = {a, b, c, d}

35. *S* = {l,2, 3, 4}, *T* = {l, 3, 2, 4}

36. *B* = set of letters in the word "pool;' *C* = set of letters in the word "lop"

37. *C* = the set of U.S. state capital cities, *S* = the set of U.S. states

38. *A* = $\{x \in N \mid x > 2\}$, *B* = the set of numbers greater than 2