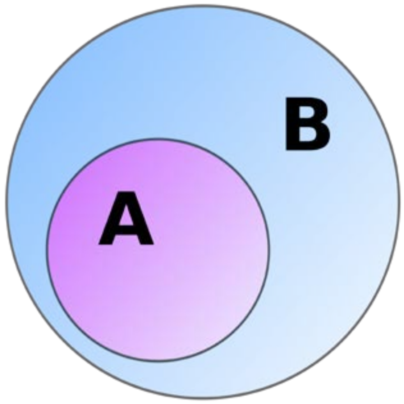

Subsets

In many situations, a particular individual may be a member of several differ-ent sets. For instance, if we consider a particular person, whom we can call John Smith, it is certainly true that John is a member of his own family, but his family is, in turn, a member of a larger group, such as a neighborhood community. That community is part of a town, which is part of a county, which is part of a state, and so on. This illustrates that some sets happen to be contained within other sets.

Given two sets, A and B, suppose that every element of A is also an element of B. This condition gives a relationship between the two sets, called the **subset** relationship. This is denoted by the symbol ⊆ , which is read as "is a subset of' In symbols, $A \subseteq B$ if $a \in A$ implies that $a \in B$.

Every nonempty set must have at least, trivially, two subsets. The set itself satisfies the definition of a subset, so for all sets A, we must have $A \subseteq A$. Ad-ditionally, the empty set is a subset of any set. That this is true is a consequence of the definition of the subset relationship. There are no members of the empty set that are not members of A, so Ø is a subset of A.

1. Find all subsets of the set {1,2,3,4} which have cardinality three. (Note: Don't forget, a subset is a set itself and so its elements should be contained by braces.)

2. Find all subsets of the set {1,2,3,4} which have cardinality two.

3. Find all subsets of the set {1,2,3,4} which have cardinality one.

Proper Subsets

If *A* is a subset of *B,* then *B* is called a **superset** of *A,* and if the set *B* contains additional elements that are not elements of *A,* then we say that *A* is a **proper subset** of *B* and use the specific notation $A \subset B$. When it is unclear whether *A* is a proper subset of *B* or if we prefer to maintain generality, the regular subset symbol, $\subseteq$, may be used, but if it is known that the subset relationship is proper, use the proper subset symbol.

Referring back to our known sets of numbers from mathematics, we can establish a chain of subsets as an illustration: $N \subset W \subset Z \subset \mathbb{Q} \subset R$. Note that each of the subset relations in the example is a proper subset relation, since each superset contains additional members beyond those of the subset.

4. For each of the following pairs of sets A and B, determine whether set A is a subset of set B.
   a) A = {John, Mary, Bob}, B = {John, Mary, Bob, Scott}

   b) A = {2, 3, 5, 7, 11, 13, ...}, B = N

   c) A = set of states in the United States, B = {New York, New Jersey}

5. Given *A* = {a} and B = {a,b, c}. State the subset relationship between sets *A* and *B*.

The Power Set of a Set

Let's consider a particular example of a relatively small set, A = {l,2, 3}, and at-tempt to list all its subsets. To begin, we have already stated that, by assumption, Ø is a subset of A and that A is a subset of itself. Now let's turn to the other subsets. First, consider the one-element subsets. A set with one element is called a **singleton set**. There are three such subsets of A: {1}, {2}, and {3}. Now consider the two-element subsets: {1, 2}, {1, 3}, and {2, 3}. We have completed the list of all possible subsets of A and have found that there are a total of eight: *A*, Ø, {1}, {2}, {3}, {1, 2}, {1,3}, and {2, 3}.

The set we can make whose members are all the subsets of a set A is called the **power set** of A, denoted 2^A or sometimes *P(A)*. Thus, another way of asking you to list all the subsets of a set A is to call for the power set of A.

It would be useful for us to know how many subsets a particular set would have so that we would know for sure we had listed them all. It turns out that for finite sets of cardinality *n*, there are always 2^n subsets. Note that a set with just one element satisfies this rule, since the set itself and the empty set are its only subsets and that the empty set (which has only itself as a subset) also satisfies the rule, since $2^0 = 1$. For even relatively small sets, the number of subsets can be quite alarming.

Consider the set of letters in the English alphabet, which has 26 members. That set has more than 67 million subsets! If you could write one subset per second, working continuously and not pausing for the inconvenience of sleep, it would take you *more than two years* to produce the complete listing.

Just as we had a symbol to represent nonmembership in a set, we have a symbol to indicate the nonexistence of the subset relationship. The symbol is the same as the subset symbol but with a slash through it: $\not\subset$. This symbol is read as "is not a subset of'

6. Let S = {1,2,3,4}, as above. Use your results above to write out the power set P(S) in its entirety. (Hint: There should be 16 elements, each of them a set in its own right.)

7. Suppose now we removed the element 4 from the set S. I.e., suppose S = {1,2,3}. Write out the new power set 2^S.

8. What is the cardinality of the power set 2^S in Investigation 7?

9. Now suppose we have also removed the element 3 so S = {1,2}. Write out the power set 2^S.

10. What is the cardinality of the power set P(S) in Investigation 9?

11. Suppose finally that we removed the element 2 as well so S = {1}. Write out the power set P(S).

12. What is the cardinality of the power set 2^S in Investigation 11?

13. Using the results of these investigations, how does the cardinality of a finite set S appear to be related to the cardinality of its power set 2^S?

14. The pattern you described in Investigation 13 continues indefinitely. Explain why it is appropriate to write

$$\text{Card}(2^S) = 2\text{Card}(S).$$

15. For the following sets, determine if A ⊂ B, B ⊂ A, $A \not\subset B$, $B \not\subset A$, A = B, or if none of these relationships exist. it is possible that more than one relationship could hold, and if that is the case, list all applicable relationships.

a. A = {a, e, i, o, u}, B = set of letters in the English alphabet

b. *A* = set of positive even integers less than 20, *B* = {O, 2, 4, ..., 20}

c. $A = \{x \mid x \in N, 10 < x < 20\}$, $B = \{x \mid x \in N, 11 \leq x \leq 19\}$

d. *A* = { x I x is a retired professional basketball player},
B = {Michael Jordan, Larry Bird, Julius Erving}

e. *A* = players in the starting lineup for the NY Yankees in the first game of the 2009 World Series,
B = players in the starting lineup for the Philadelphia Phillies in the first game of the 2009 World Series

f. *A* = Ø, $B = \{x \mid x \in N, x < 5\}$

g. *A* = set of letters in the word "Mississippi;' *B* = set of letters in the word "sin"

Cantor's Theorem and Transfinite Cardinal Numbers

Numbers denoting the cardinality of sets are called *cardinal numbers*. We know there are sets of cardinality 1,2,3,.... Cantor's theorem shows us that there is an entire hierarchy of cardinal infinities beyond the finite cardinal numbers. His theorem is as follows:

> **Cantor's Theorem**: Given any sets S, the power set P(S) has a strictly larger cardinality.

In light of what you saw in Investigation 14, Cantor's theorem does not seem surprising. However, Cantor's theorem applies to infinite sets as well and thus guarantees that there are strictly larger sizes of each infinity!

To give it a symbolic name, we denote the cardinality of the natural numbers by $\aleph_0$ = Card(N). Here $\aleph$ is aleph, the first letter in the Hebrew alphabet. The subscript 0 is because this is the first, or smallest, infinite cardinal number.

16. Explain why the cardinality of the set of squares {1,4,9,16,...} is $\aleph_0$.

17. Explain, directly without the use of Cantor's theorem, why the power set of the natural numbers, P(N), must be an infinite set.

18. Mathematicians generally refer to the cardinality of P(N) by $2^{\aleph_0}$. Explain why.

19. What does Cantor's theorem tell you about the relative sizes of the two infinite cardinal numbers $\aleph_0$ and $2^{\aleph_0}$?

20. Because P(N) is itself a set, we can form its power set - P(P(N)). Why is $2^{\aleph_0}$ a good name for the cardinality of this set of sets?

21. What does Cantor's theorem tell you about the relative sizes of the cardinal numbers $\aleph_0$, $2^{\aleph_0}$, and $2^{2^{\aleph_0}}$?

22. Explain how this process can be extended indefinitely to find an infinite number of different sizes of infinite cardinals.

As we describe in the Interlude, Cantor was a deeply religious man who was attacked by many for his exploration of the infinite. Throughout his groundbreaking work Cantor clearly distinguished between what he referred to as transfinites and what others thought of as completed infinities. Cantor's defense, in his own words, is:

I have never proceeded from any "Genus supremum" of the actual infinite. Quite the contrary, I have rigorously proven that there is absolutely no "Genus supremum" of the actual infinite. What surpasses all that is finite and transfinite is no "Genus"; it is the single, completely individual unity in which everything is included ...which by many is called "God."[1]

23. Explain how your result in Investigation 22 is related to Cantor's quote about God.

24. Find and then briefly describe several other important mathematical and/or scientific developments whose proponents were attacked because their views did not conform to prevailing religious views.

How might we judge these controversies in hindsight given the benefits of a great many years experience?

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[1] Quoted in *Georg Cantor: His Mathematics and Philosophy of the Infinite* by Joseph Warren Dauben, p. 290.