Hilbert’s Hotel

The development in the previous lesson of an infinite heirarchy of infinities of increasing size relied on Cantor’s theorem and building sets of sets. How, you may ask, do we know that Cantor’s theorem is true? At a major conference in 1999, the author gave a talk entitled “Cantor’s Theorem: Mathematics’ Most Accessible Masterwork.” The proof of Cantor’s theorem, which included clarifying examples, fit on less than one-half of a single overhead slide which used 36 point font![1] Alas, if you find sets of sets a bit abstract, this proof is not entirely enlightening. If you feel this way a natural question to ask is:

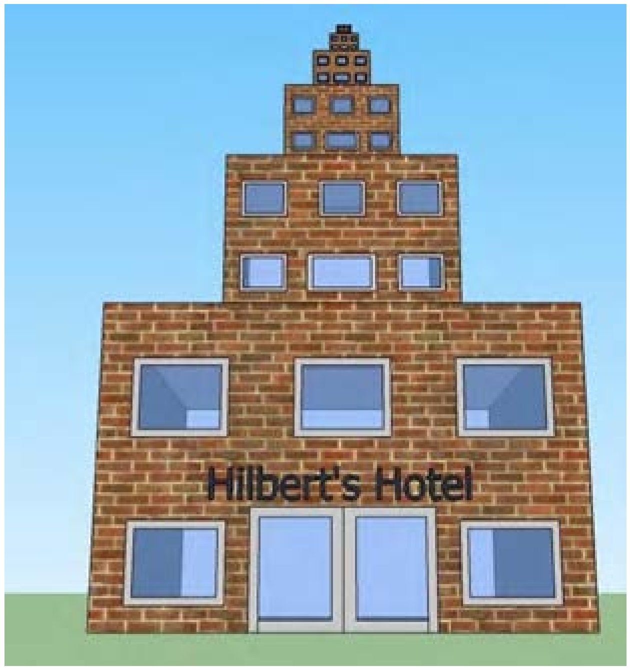
> If there are infinitely many different sizes of infinity, can’t you just show us an example of two different sizes?

Good question. This is where we now turn.

One way of describing one-to-one correspondences between certain infinite sets has become known as **Hilbert’s Hotel** after David Hilbert (German mathematician; 1862 - 1943).[2]

You really need to use your imagination to conceptualize Hilbert’s hotel. Several people have written science fiction about this hotel. Here’s our version:
You awake, groggy and a bit disoriented. You walk around but do not recognize much. Up the street you see a strange but otherwise unimposing building which appears to be called Hilbert’s hotel. Your view of it looks like the image in Figure below.

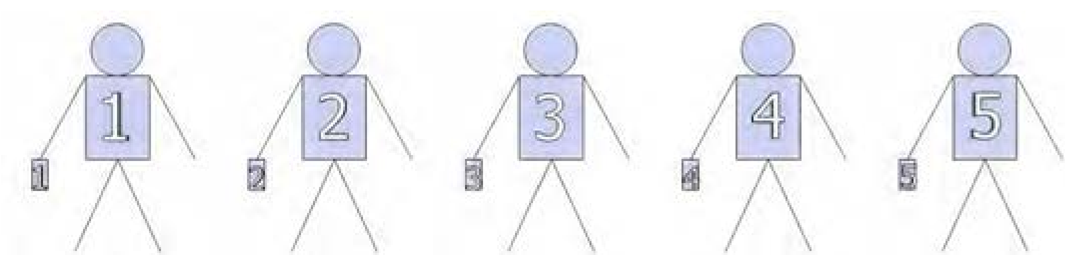
[1] For a similar example in print, see p. 50 of Mathematics: A New Golden Age by Keith Devlin.
[2] In One, Two, Three ...Infinity George Gamow (; -) attributes this description to “the unpublished, and even never written, but widely circulating volumes: “The Complete Collection of Hilbert Stories” by Richard Courant (German American mathematician;1888 - 1972).”



At first the building seems very tall, like you are viewing it from its base in an exaggerated perspective. But you realize that the floors are actually getting smaller very quickly.

Nonetheless, it looks harmless enough, so you go in.

It seems nice. You notice a very big group of alien looking creatures. Each wears a number on its chest, almost like a nametag, and holds in their hand a small card which is similarly numbered. They look the creatures below.



Some of the natural number guests at Hilbert's hotel

You approach the desk, hoping there is an available room where you can regroup and try to figure out what's going on.
"Do you have any available rooms," you ask.

The clerk responds, "No, we are completely full. The natural numbers have the entire hotel booked." You pause, trying to figure out your next move. "Would you like to stay?" "I thought you said you were completely full.” “We are, but this is Hilbert’s hotel, we have infinitely many rooms.”

You should have just accepted the offer, but you can’t help asking, “How can you have infinitely many rooms?” “We have 20 rooms on each layer and infinitely many layers” the clerk says nonchalantly. You think back to your strange view of the hotel’s facade. “Do you want a room or not?” the clerk asks. “Yes, please.” “Do you mind being scaled?” “Yes, scaled. Compressed. Reduced. Shrunk. Even though it is perfect harmless, some people object to this. So we put them in the lower layers. If you’re on the second layer the elevator only scales you to have your size. On the third layer you’re scaled to one-quarter of your size.” The clerk senses your confusion. “I’ll just put you in Room 1 right here in the first layer so you are comfortable.”

“Hey 1,” the clerk yells. “We need to accommodate this newbie. Can you please get your group to all move down a room?” “Certainly,” 1 responds. S/he hands you the card imprinted with a 1. You realize that it is a key card for your room, Room 1. You see 1 return to the group. 2 hands 1 the keycard for Room 1, 3 hands 2 the keycard for Room 2, and so on. 20 gives up the keycard for Room 20 and heads off to the elevator. “Where’s 20 going?” you wonder, feeling badly that you’ve kicked somebody out of their room. “Up to level 2. Room 21 will now be 20’s room and the other guests will also move down. 999,999,999 will be really happy to have the Million Room.”

1. Describe all guests that have to move to another layer due to your arrival.

2. Suppose another guest arrived looking for a room. Could they be accommodated? How?

You sleep soundly and as you awake you hope that you are back in your own world. As you adjust to the rooms low light you realize you are still at Hilbert’s hotel. After a shower you get dressed and head to the lobby to find some food.

As you approach the desk a different clerk yells at you, “You’re late. Where have you been, we’re swamped? Here’s your jacket.” You take the jacket which you see has your name on the left and “maitre d” on the left.

People - of the alien number sort that you saw yesterday - are clamoring for rooms. You see 0,−1,−2,−3,−4 and −5 and a long line leading out the door. “Are all of the negative integers here?” you wonder.

Quickly you look at the registry from two nights before. It looks as follows:

Room Number	Guest Name
1	1
2	4
3	9
4	16
5	25
$\vdots$	$\vdots$

3. With the registry above, did the “squares” fill every room of Hilbert’s hotel? Explain. The positive integers {1,2,3,...} are staying another night. How can you accomodate 0 and the negative integer s $\{\ldots,-3,-2,-1\}$. (Note: And there is no room - sharing allowed! Hint: What happens if you move 1 to Room 4 and 2 to Room 6?)

4. Later in your shift the phone rings. The caller asks if there is a vacancy for Sunday night. You look in the registry and see that no rooms are booked. “How many in your party?” you ask. The caller responds “$\aleph_0$.” Can the group be accommodated? If so, explain precisely how you can assign rooms.

There are many other important sets that can be accommodated by Hilbert’s Hotel - including those we might think would be too big. For example, the set of all fractions (a.k.a., the rational numbers) can be accommodated; they have cardinality $\aleph_0$. Even the set of solutions of every algebraic equation (which are called the algebraic numbers) can be accommodated. They too have cardinality $\aleph_0$.

Nondenumerability of the Continuum

What Cantor showed in his groundbreaking 1873[3] discovery, was that the set of all points that make up the unit interval of the real number line, [0,1], cannot be accommodated by Hilbert’s Hotel. This set of points is too big. In other words, the cardinality of the unit interval [0,1], which is usually denoted by Card([0,1]) = c for continuum, is strictly greater than $\aleph_0$! In other words, a set that we know intimately has a higher cardinality than $\aleph_0$.

To show that the points in the unit interval cannot be accommodated by Hilbert’s Hotel we need to show that every possible Hilbert’s Hotel registry will leave out at least one point in the unit interval. To get a sense of what this entails, we are returning you to your role of maˆtre d of́ Hilbert’s Hotel.

[3] See, e.g., p. 50 of Georg Cantor: His Mathematics and Philosophy of the Infinite by Joseph Warren Dauben.

Room Assignment	Guest
1	0. _ _ _ _ _ _ _ _ _ _ ...
2	0. _ _ _ _ _ _ _ _ _ _ ...
3	0. _ _ _ _ _ _ _ _ _ _ ...
4	0. _ _ _ _ _ _ _ _ _ _ ...
5	0. _ _ _ _ _ _ _ _ _ _ ...
6	0. _ _ _ _ _ _ _ _ _ _ ...
7	0. _ _ _ _ _ _ _ _ _ _ ...
8	0. _ _ _ _ _ _ _ _ _ _ ...
⋮	⋮

5. Write down 20 different points that belong to the unit interval [0,1].

6. Describe the points in [0,1]. What can you say about them as decimals?

7. Assign some point in [0,1] to Room 1 in the registry above by writing its decimal digits in the appropriate places. If the decimal is finite, after it terminates write zeroes to the right of the final decimal digit as in 0.7168366610 = 0.716836661000... If the decimal is infinite, give at least the first ten digits.

8. Assign some other point in [0,1] to Room 2 in the registry above, as you did in Investigation 7.

9. Assign some remaining point from [0,1] to Room 3 in the registry above, as you did in Investigation 7.

10. Assign some remaining point from [0,1] to Room 4 in the registry above, as you did in Investigation 7.

11. Repeat Investigation 10four more times.

12. If you continued to assign rooms like this, would there ever be a point at which every point in [0,1] had been assigned a room? Explain.

13. Suppose you were able to assign rooms faster and faster so that in a finite amount of time all infinitely many rooms in Hilbert’s Hotel had been assigned - almost in reverse of how we can add up infinitely many terms of a series and get a finite sum. Do you think that then all of the points in [0,1] will have been accommodated? Explain.

It is my sad duty to inform you that you that even if you have completed your entire registry, you have not assigned rooms for all members of [0,1]. I know this because you are going to help me find a number of [0,1] that you have not assigned a room to.

We're going to find this number one decimal at a time. We will represent this number by the decimal $0.d_1d_2d_3d_4 \ldots$

14. Circle the first decimal digit of the guest you have assigned to Room 1. If this digit is a 1, then define $d_1 = 3$. If it isn't a 1, then define $d_1 = 1$.

15. Circle the second decimal digit of the guest you have assigned to Room 2. If this digit is a 1, then define $d_2 = 3$. If it isn't a 1, then define $d_2 = 1$.

16. Circle the third decimal digit of the guest you have assigned to Room 2. If this digit is a 1, then define $d_3 = 3$. If it isn't a 1, then define $d_3 = 1$.

17. Continue this process to define the digits d_4,d_5,d_6,d_7 and d_8.

18. If you had assigned rooms for more points in [0,1] could you continue to assign decimal digits d_n in this same way? For how long?

19. Now write down the number $0.d_1d_2d_3d_4 \ldots$ Does this decimal represent a unique, well-defined real number in the unit interval [0,1]? Explain.

20. Explain why $0.d_1d_2d_3d_4 \ldots$ is not the guest in Room 1.

21. Explain why $0.d_1d_2d_3d_4 \ldots$ is not the guest in Room 2.

22. Explain why $0.d_1d_2d_3d_4 \ldots$ is not the guest in any of the Rooms 3 - 8.

23. Explain how I know that $0.d_1d_2d_3d_4 \ldots$ will not be in Room 37.

24. Explain how I know that $0.d_1d_2d_3d_4 \ldots$ has not been assigned to any of the rooms that you assigned - even if you have assigned every room in Hilbert's hotel.

You have failed in your task of assigning all points in [0,1] a room. But so did your peers. This same rule that I used to find a point that you did not assign a room to can be used to show that your peers have missed points as well. Their roomless guest points will likely be different, but they will be found in the same way. In other words, this rule will find a point that is missing from any possible guest registry for Hilbert's Hotel.
From this failure we can only conclude that no such guest registry is possible; i.e. that the points in [0,1] are too numerous to fit in Hilbert's Hotel. This means that the cardinality c is a different cardinality than $\aleph_0$. We now have two different sizes of infinity.

The Continuum Hypothesis

Cantor rapidly developed the hierarchy of infinite cardinals that begins as we have described above as a result of Cantor's theorem. As we shall discover in the next chapter he also developed an infinite hierarchy of infinite ordinals. And he developed a beautiful relationships between these two different conceptions of quantifiable infinite.

Yet he spent much of the latter part of his life vexed by a single question: are there sets whose cardinality is between $\aleph_0$ and c? Cantor believed that there was no such set - that the infinite cardinal c followed the first infinite cardinal $\aleph_0$ immediately. This result became known as the continuum hypothesis. It is likely that his inability to solve this problem, coupled with his contentious relationships with other prominent mathematicians of the day about matters involving the infinite, contributed to his declining mental status. So important is this problem that it was the first of Hilbert's problems, a list of 23 problems posed in 1900 that were to focus mathematical research through the twentieth century.

Cantor died before this problem was resolved.

But Cantor had the last laugh. Beginning with the most widely used basis for set theory Kurt Gödel (Austrian and American, logician, mathematician, and philosopher; 1906 - 1978) proved in 1938 that the continuum hypothesis is consistent with the axioms; i.e. it cannot be disproved. But, decade after decade, no proof was forthcoming. In 1963, Paul Cohen (American mathematician; 1937 - 2007) proved that the continuum hypothesis cannot be disproved with these same axioms. In other words, these two mathematicians proved that Cantor's continuum hypothesis was undecidable.[11] Cantor did not know the answer; nobody else will either.

25. **Classroom Discussion**: In writing about Cantor's proof of the uncountability of the continuum, William Dunham (American mathematician and author; 1947 -) wrote:

Any layman who, in 1874, had visited Paris to see the paintings of Claude Monet would have been struck by the "impressionist" techniques the artist had introduced. Even the casual observer would have seen in Monet's brushwork, in his rendering of light, a significant departure from the canvases for such predecessors as Delacroix or Ingres. Clearly something radical was going on. Yet in this mathematical landmark of the same year, Georg Cantor had set upon a course every bit as revolutionary. It is just that mathematics on the printed page often lacks the immediate impact of a radical piece of art.

26. You have now had the opportunity to work with this radical idea that there are different sizes of infinity. You have "seen" it with your own intellect. What do you think? Is this piece of work worthy of the title "revolutionary"? Is this work beautiful? Do you see how it may have a lasting impact on all future considerations of the infinite?