Establishing Truth

In this chapter we would like to consider the ways in which we establish truth. We believe that by considering the ways in which we seek to establish truth in several different real-world settings will help illuminate aspects of mathematics that are often not understood. In later chapters we will see that when we proceed to analyze the foundations of mathematics we will be repaid with far reaching implications to the nature of truth and certainty.

The U.S. Courts System - A Case Study

O.J. Simpson (American Athlete and Actor; 1947 -) was the first professional football (American) player to rush for more than 2,000 yards in a single season. In addition to stellar collegiate and professional athletic careers he was a successful actor and television salesperson. His personal life was much less successful.

On 12 June, 1994 Simpson’s ex-wife Nicole Brown (American Housewife; 1959 - 1994) and her friend Ronald Graham (American Model and Waiter; 1968 - 1994) were brutally murdered. Simpson became a suspect and was to be charged before he attempted to flee in a slow-speed chase televised live on 17 June, 1994. This began a highly publicized, sensationalized, divisive, and racially charged series of trials.

The outcome of the trials were that Simpson was acquitted of double murder charges while he was found responsible for the wrongful deaths of Brown and Graham. How someone can be tried twice with opposite results is confusing to some. The reason we describe this case here is that it can help us understand different standards of proof.

	Civil Court	**Criminal Court**

Decision	Whether one individual has harmed another	Whether an individual has broken a law
Parties	A *plaintiff* brings the case against a *defendant*	A government *prosecutor* brings the case against a *defendant*
Questions	Was there *damage*? Who is *responsible* for the damage?	Was a *crime* committed? Who *committed* the crime?
Finding/ Conclusion	*Responsible*, or not, for damage	*Guilty* or *innocent* of crime
Outcome of Positive Finding	The *remedy* is *damages* which are often monetary	The *penalty* is incarceration or other deterrent
Burden of Proof	*Perponderence of evidence*	*Beyond a reasonable doubt*

Notice all of the technical legal terms that have been used without being given formal definitions.
The definitions of these terms are determined by statute, by legal precedence, and by court inter-
pretation. These terms, and those like them that make up the legal vernacular, have very specific, precise definitions despite the fact that these definitions evolve over time.

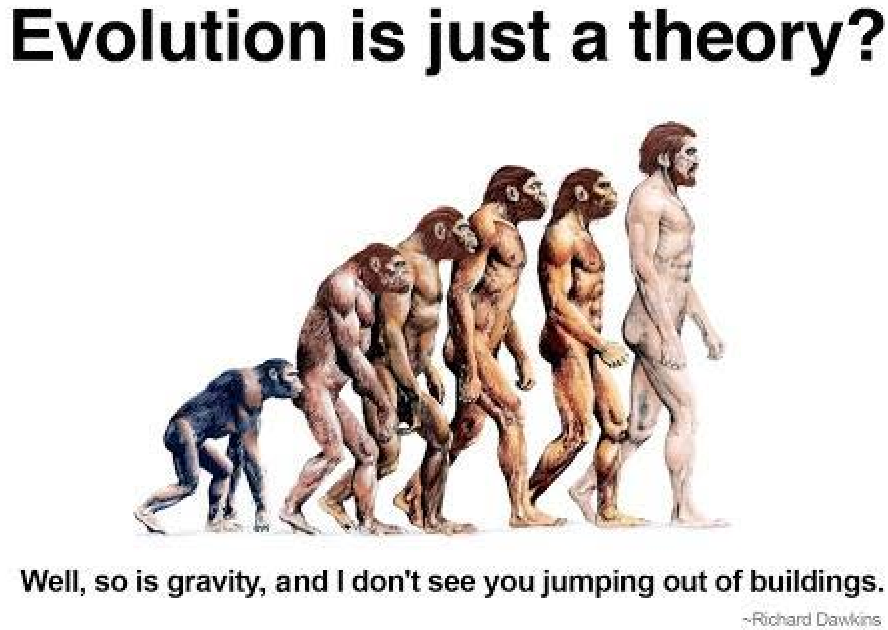
So what is the truth? Did Simpson brutally murder Brown and Goldman? Only a few people actually know. But our court system has been developed so we have a structure, however fallible, that enables us to try to find relief for damages and crimes.


INSTRUMENT OF THE RICH
THE ONION
PACIFIER OF THE POOR
Monday, July 20, 1925 — Finest Source of News — In Our Great Republic — Price Three Cents
SCOPES MONKEY TRIAL RAISES TROUBLING QUESTION:
IS SCIENCE BEING TAUGHT IN OUR SCHOOLS?
Scopes Trial Called 'Trial of the Century'
DAYTON, Tenn., July 19 —The sensational trial of science teacher John Scopes is now being called the "trial of the century" by experts in the study of law. Never before—and likely
SHOULD CHILDREN BE EXPOSED TO FACTS?
Are Reason and Empirical Evidence Suitable School Subjects?
The Opinion of Edward Tillich, The Onion's Crime and Science Editor
and coal miners, facts are the last things they will need. Manners! Subservience! Above all, obedience!
Scopes Defended By Super-Intelligent Chimpanzee-Man from Future
Cornelius, a beast imbued with speech and learning
William Jennings Bryan

Science and Religion

The Scopes Monkey Trial was a 1925 case in which charges were brought against **John Scopes** (American Teacher and Geologist; 1900 - 1970) for teaching the theory of evolution in a Tennessee school. This case is one of the more well-known cases in American legal history. It was widely followed as it happened, providing a huge stage for the opposing lawyers **Clarence Darrow** (American Lawyer; 1857 - 1938) and **William Jennings Bryant** (American Politician; 1860 - 1925) to joust over the political, legal, and religious implications. Scopes was convicted and his conviction was later overturned on appeal.

The debate over the teaching of evolution in the schools continues to this day. While this is not a place to address this topic in general, it does bear relevance to our effort to understand how we establish truth.



The implicit suggestion of the graphic above is that if we are to question evolution because it is "just a theory", we might as well question gravitation as well. It certainly is just a theory. But, like evolution, it is a profoundly useful theory that has been justified by hundreds of thousands of experiments over centuries. We consider the "theory of gravitation" to be true without giving it much though, but continue to question the "theory of evolution." It is of interest that in modern physics, gravity is no longer considered a fundamental force. Einstein's Theory of Relativity (yes, another theory) purports to explain what we have thought of as gravity is actually travel along *geodesics* in *spacetime* whose "lines" have been curved by the presence of matter. In this context, gravity appears part of an exotic geometry rather than a truth of nature.

More generally, the ongoing debate between religious and scientific views is based largely on the different paradigms that one accepts/believes when one searches for truth.

Who Are You Going to Believe

The number π is perhaps the most fundamental constant in mathematics. Approximated to six decimals places as $\pi \approx 3.1415926$, the number π is what mathematicians call a transcendental irrational. Its decimal representation goes on indefinitely without any repeating pattern ever appearing. π is not the solution to any algebraic equation with rational number coefficients.

Despite these difficulties, π is central to mathematics. Most basically, it is the constant that relates linear dimensions (e.g. the radii, diameters, and heights) of circles, spheres, cylinders, cones and other objects which fundamentally involve circles to the areas, surface areas, and volumes of these objects. These relationships were known to the ancients even though their knowledge of numerical approximations to the actual value of π progressed very slowly over the millennia.

In the American King James version of the Bible, 1 Kings 7 describes Solomon’s temple. Solomon fetches Hiram of Tyre who forges a hemispherical brass tub. In 1 Kings 7:23 the size of this hemisphere is described as follows:

its lip	to	from its lip	by the cubit	ten	casted	the sea	And he made
שפתו	עד-	משפתו	באמה	עשר	מוצק	את-הים	ויעש

by the cubit	thirty	and a line	its height	by the cubit	and five	all about	round
באמה	שלשים	וקוה	קומתו	באמה	וחמש	סביב	עגל

all about	it	went around
סביב:	אתו	יסב

And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.

1. Draw a hemisphere and carefully label all of the dimensions that are given in the passage from 1 Kings.

2. What part of the hemisphere is being measured as “thirty cubics did compass it round about”? What is the precise mathematical term that describes the measurement of this part?

3. Find a formula which expresses the measurement in Investigation 2 as a function of linear dimensions of a hemisphere.

4. Pretend that you did not know the value of the mathematical constant π. Use the equation in Investigation 3 and the quantities given in 1 Kings to find a Biblical Value of π. Does this bother you? Explain.

Bill #246 of the 1897 Indiana General Assembly was:

> A Bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the Legislature of 1897.

The bill was brought for legislative action by Taylor I. Record (American Indiana State Representative) on behalf of Edwin J. Goodwin (American physician; ca. 1825 - 1902). In the bill Goodwin claims to have found solutions to the Three Construction Problems of Antiquity which have been proven to be impossible to solve:

- Squaring the Circle
- Trisecting an Angle
- Duplicating a Cube

5. One of the “facts” that would have been legislated by the bill is the legitimacy of a circle with radius 5 and circumference of 32. Find a formula which expresses the relationship between the circumference and radius of a circle.

6. Pretend that you did not know the value of the mathematical constant π. Use the equation and data in Investigation 5 to determine the numerical value of π that would have resulted if this legislative act was passed.

The Indiana House Committee on Education recommended that the bill be passed. Subsequently, it was approved the the Indiana House of Representatives unanimously 67 to 0! Luckily Clarence Abiathar Waldo (American mathematician and educator; 1852 - 1926), then Chair of the Purdue
University Department of Mathematics, was in attendance when the bill was taken up by the Indiana Senate. At the General Assembly to lobby for legislative support for the University he reverted to his role as teacher, educating a sufficient number of Senators that the bill was indefinitely postponed.

7. **Classroom Discussion**: The title of this activity is “Who are you going to believe?” Have these examples changed and/or informed the way that you think of the fiats of “authorities”? Can you think of other, similar examples?

Of course, these examples have a certain definiteness because they involve fairly clear statements about a well-established mathematical constant. But the implications are far broader. For example, the distinguished Peter J. Gomes (American Preacher and Theologian; 1942 - 2011), who served as Minister and Plummer Professor of Christian Morals at Harvard

from 1974 until his death, spoke eloquently about “reading the Bible with mind and heart.” He wrote:
The last thing the faithful wish for is to be disturbed. Thus it is easy to favor the Bible over the gospel...Could it be that we spend so much time trying to make sense of the Bible, or making it conform to our set of social expectations, that we have failed to take to heart the essential content of the preaching and teaching of Jesus? ...If we are sincere in wanting to know what Jesus would do, we must risk the courage to ask what he says, what he asks, and what he demands. Only if we do so will we be able to move, however cautiously and imperfectly, from the Bible to the gospel.[1]

8. **Classroom Discussion**: From whom does the authority of the Bible come from? Does the answer depend on whether we approach the question from historical, political, religious, or personal perspectives? In Part 2, “The Uses and Abuses of the Bible,” of The Good Book: Reading the Bible with Mind and Heart

Rev. Gomes discusses in great length what the Bible tells us about race, what it tells us about Semites, what it tells us about women, and what it tells us about homosexuality. What is the difference between the Bible and the gospel that Gomes seeks so carefully to distinguish? What impacts might these distinctions have on what we choose to believe is moral?

8. **Classroom Discussion**: The Pulitzer- and Tony-Award winning play *Proof* was the first full-length work by playwright David Auburn (American playwright; 1969 -). Lead character Catherine is the daughter of the crazy mathematician Robert, Hal the diligent graduate student. The passage below is fundamental to the d´enouement:

> Catherine: You think you’ve figured something out? You run over here so pleased with yourself because you changed your mind. Now you’re certain. You’re so... *sloppy*. You don’t know anything. The book, the math, the dates, the writing, all that stuff you decided with your buddies, it’s just evidence. It doesn’t finish the job. It doesn’t prove anything.
>
> Hal: Okay, what would?
>
> Catherine: *Nothing*. You should have trusted me.

How do burdens of proof differ in real life? In love, medicine, religion, friendships, media, politics,...?

[1] From The Scandalous Gospel of Jesus: What’s So Good About the Good News?, pp. 19,23.

The Scientific Revolution

This is not a book about faith. We hope to illustrate different standards of proof. In doing so, we need to consider the context of the Scientific Method that is central to scientific reasoning. From The Scandalous Gospel of Jesus: What's So Good About the Good News?, pp. 19,23.

The terms Renaissance, Age of Enlightenment, and Scientific Revolution are terms invented by contemporary historians to describe what they see as important periods in European culture.

8. In your own words, briefly describe the Renaissance.

9. Cite a few mathematical and/or scientific advances that played important roles in the Renaissance.

10. In your own words, briefly describe the Age of Enlightenment.

11. Cite a few mathematical and/or scientific advances that played important roles in the Age of Enlightenment.

12. In your own words, briefly describe the Scientific Revolution.

13. Cite a few mathematical and/or scientific advances that played important roles in the Scientific Revolution.

14. Which of these periods is subsumed by the others?

Fundamental to these periods was the development of the *Scientific Method*.

15. In your own words, describe the Scientific Method.

16. Illustrate, using a few different examples of paradigms that you considered above, how the scientific method was employed to result in important paradigm shifts.

Mathematics and Deductive Reasoning

> Faith is different from proof; the latter is human, the former is a Gift from God.
>
> Blaise Pascal (French mathematician, physicist, inventor, writer and Christian philosopher;1623 - 1662)

The Scientific Revolution is considered one of the most important intellectual developments in history. The resulting world-view has dominated science, and the social sciences who have sought to mimic its methods in many ways, for centuries.

Much less well-known, but equally important, is the revolution that formed the *epistemological* basis for mathematics. This revolution took place approximately two millennia before the Scientific Revolution. Here we consider this revolution from a general, theoretical perspective. In the next chapter it is explored in more practical, concrete ways.

Aristotle (Greek philosopher; 384 BC - 322 BC) was the first we know to undertake a detailed study of what we now call logic. Central to our purpose was his effort to illustrate what we now call deductive reasoning. In *Prior Analytics* he writes, "...certain things being supposed, something different from the things supposed results of necessity because these things are so." **Deductive reasoning** is the method of reasoning which links premises and axioms to conclusions - the truth of the premises and axioms logically necessitating the truth of the conclusion.

If conclusions are built on premises, one may fairly wonder how the premises were derived. For concreteness, let us consider one of the earliest, large-scale use of deductive reasoning: Euclid's *Elements*. In 13 books, Euclid deductively establishes 468 propositions. Upon what do they rest? Euclid begins with 23 definitions. These are *primitive terms* which are taken to need no further articulation. For example, Definition 5 states: A **straight line** is a line which lies evenly with the points on itself. He has 5 common notions. Among them, Common Notion 1 states: Things which equal the same thing also equal one another. Finally, he has five axioms or postulates, which are the unquestioned assumptions or building blocks for the system of Euclidean geometry he is setting out to create. Axiom 1 states: To draw a straight line from any point to any point. This enables the use of a straightedge in this geometry. Axiom 3 states: To describe a circle with any center and radius. This enables the use of a compass in this geometry. Euclid is now free to build a world where conclusions follow by logical necessity from the axioms and common notions. His first result is:

Proposition I.1 An equilateral triangle can be constructed from a given straight line.

19. Draw a straight line segment.

20. Proposition 1 says that an equilateral triangle, a triangle with all sides congruent, can be constructed from this straight line. Use the tools given by Axioms 1 and 3 - a straightedge and compass - to construct an equilateral triangle.

21. Prove that your triangle is equilateral. (Question: How does Common Notion 1 play into your proof?)

It is essential to note that it is impossible to refute this proposition if one starts from the same axioms and common notions. The truth, built from first principles, with necessity required at each stage of building, is eternal within the system that Euclid set out. This is essential for it has enabled mathematics to be “more successful than other branches of human knowledge in its endeavor to erect a reliable and lasting structure of thought.”[2] It is lasting because:

> In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure.
>
> Herman Henkel (German mathematician; 1839 - 1873)

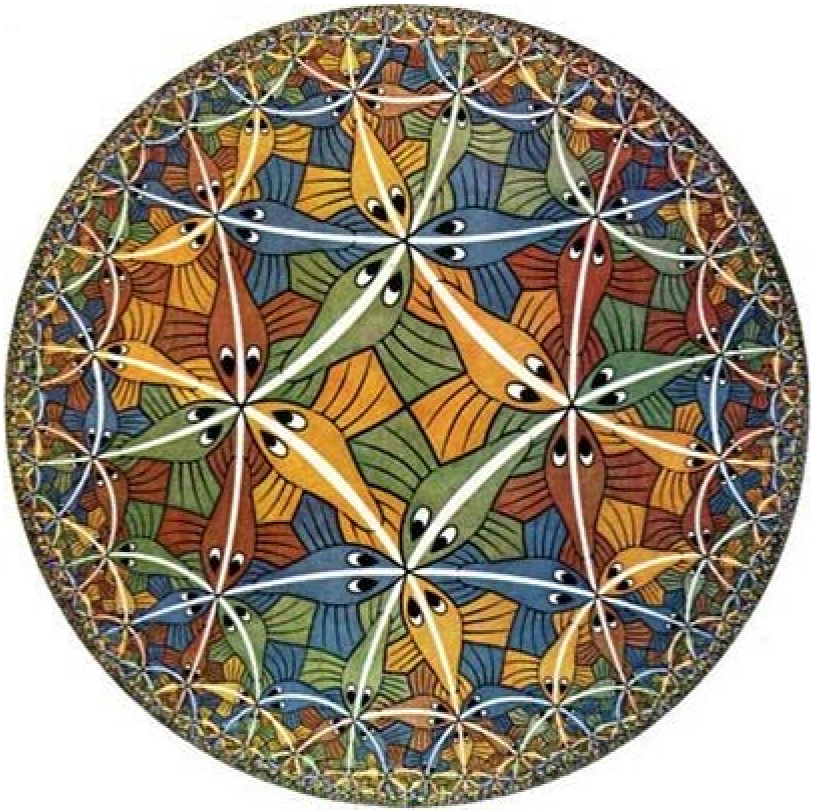
One may only question results of deductive system by questioning the foundational postulates of the system. The remainder of the structure has been built as logical consequences from this foundation. It is for this reason that “each generation adds a new story to the old structure.” What happens when one questions or disputes any part of the foundation of the system? While much of the growth of mathematics builds vertically on established systems, important advances have come from beginning anew. By removing just one of Euclid’s postulates - the 5th postulate which is often called the parallel postulate - geometer’s of the nineteenth century created entirely new, important geometries. **J´anos Bolyai** (Hungarian mathematician; 1802 - 1860) said of these geometries:

> I have discovered such wonderful things that I was amazed... Out of nothing I have created a strange new universe.

These new geometries did not discredit or invalidate Euclidean geometry. They exist as alternatives to Euclidean geometry.

But shouldn’t many geometries pose a problem? Which is the real geometry? In the introductory case study - civil versus criminal trials in the United States court systems - we saw that there were different systems for different purposes. The same is true with the different geometries. Carpenters and engineers work in Euclidean geometry as our existence on small scales is essentially flat. Airline pilots and sea captains work in spherical geometry, following routes along the curved surface of the earth. Bolyai’s geometry, hyperbolic geometry, a model of which is illustrated in the Escher print below, is the geometry that is the basis for Einstein’s theory of relativity. Taxicab geometry, which you will meet below, is the geometry of taxicab drivers. Each of these geometries is quite useful, although quite different.

[2] In the words of the historian of mathematics Morris Kline (American mathematician and educator; 1908 - 1992).



Circle Limit III by M. C. Escher (Dutch graphic artist; 1898 - 1972)

Differential and Integral Calculus

Unlike the U.S. court system, where the burdens of proof are different, the burdens of proof in each of these geometries is equally high. They are built deductively and each is equally consistent. Each is true - what matters is which is useful in a given context.

Calculus is one of the most important areas of mathematics, a universal tool to study continuous change, rates and aggregates. It is used throughout engineering, all of the sciences, economics, and any area that studies continuously varying quantities in mathematical ways. Its roots date back to **Archimedes** (Greek mathematician, physicist, engineer, inventor, and astronomer; 287 - 212 BC) and his exploration of circles and spheres. A number of seventeenth century mathematicians made important contributions. Nonetheless, it was over a very short period that a complete theory of calculus was developed. It was developed independently by two men, **Isaac**

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[1] It is important to note, a logical development of this system is quite sophisticated. It is a structure that needs to logically support all of the calculus, including limits and completeness.

Newton (English mathematician and physicist; 1642 - 1727) and **G.W. Leibniz** (German mathematician and philosopher; 1646 - 1716). The year 1666 is known as Newton's *annus mirabilis*, or year of miracles. In his "prime of age for invention, and minded mathematics and philosophy more than at any time since" he left Cambridge to avoid the plague that had spread across many of England's urban areas to return to his family's country home in Woolsthorpe. In this year of isolation he developed calculus, developed the universal law of gravitation and made fundamental advances to what we know about optics.

Despite its remarkable success, calculus was not build on a rigorous axiomatic foundation. In studying instantaneous change, it rested on the notion of infinitesimals - infinitely small yet nonzero numbers. While most mathematicians continued blithely along buoyed by the success of the calculus, serious objections to their cavalier attitude about the subject were levied. Addressed to "infidel mathematicians", **Bishop George Berkeley** (Irish philosopher; 1685 - 1753) wrote *The Analyst* which was a scathing attack on the lack of rigor in the the foundations of calculus. The passage below is particularly interesting as it makes clear note of the logical relationships between different parts of the calculus, all pointing clearly back to the lack of substance in the nature of infinitesimals:

> It must, indeed, be acknowledged, that [Newton] used Fluxions, like the Scaffold of a building, as things to be laid aside or got rid of, as soon as finite Lines were found proportional to them. But then these finite Exponents are found by the help of Fluxions. Whatever therefore is got by such Exponents and Proportions is to be ascribed to Fluxions: which must therefore be previously understood. And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?

In his wonderful novel-like book A Tour of the Calculus, **David Berlinski** (American philosopher, educator, and author;1942 -) says:

> If the calculus is much like a cathedral, its construction the work of centuries, it remained until the nineteenth century a cathedral suspiciously suspended in midair, the thing simply hanging there, with no one absolutely convinced that one day the gorgeous and elaborate structure would not come crashing down and fracture in a thousand pieces.

Eventually gaps in the foundation arose that could not be ignored:

> The crisis struck four days before Christmas 1807. The edifice of calculus was shaken to its foundations...The nineteenth century would see ever expanding investigations into the assumptions of the calculus, an inspection and refitting of the structure from the footings to the pinnacle, so thorough a reconstruction that the calculus would be given a new name: analysis. Few of those who witnessed the incident of 1807 would have recognized mathematics as it stood 100 years later.

David Bressoud (American mathematician ;1950 -)

Our understanding of the real number line, what constitutes a function, and how to rigorously work with limits were essential to this refitting. For those with interest in learning more, Berklinsky's *A Tour of the Calculus* is wonderful and its level is compatible with the level of the book you are currently working through. Bressoud's *A Radical Approach to Real Analysis* expects a level of mathematical experience including two-semesters of calculus, but it has a beautiful historical trajectory that can be appreciated on its own.

The United States Declaration of Independence and Constitution

The United States Declaration of Independence begins:

> We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of Happiness.

\

The authors of the Declaration of Independence and the framers of the United States Constitution after them were influenced by political philosophers of the 17th century, like **John Locke** (English philosopher; 1632 - 1704), who had been profoundly influenced by the scientific revolution. The notion of self-evident truths are the *axioms* or *postulates* for a new political system that is being established.

The United States Constitution is the supreme law of the United States. It is a system based of law based on seven Articles, a Bill of Rights (the first 10 Amendments), and 17 subsequent Amendments. The framers wished to set up a system where future decisions could be based on rational thought and logic. It seems very similar to the system set out by Euclid for geometry, doesn't it?

Yet there are fundamental differences.

22. The Declaration of Independence says “all men are created equal.” Did it really mean all men? Describe several ways in which the term “men” has been reinterpreted since the time of the original writing.

23. Write a brief summary of the 1896 United States Supreme Court case Plessy vs. Ferguson.

24. Write a brief summary of the 1954 United States Supreme Court case Brown vs. Board of Education.

The cases Plessy vs. Ferguson and Brown vs. Board of Education both involved the Equal Protection Clause of the Fourteenth Amendment. While the Amendment itself had not changed, the Supreme Court ruling was exactly the opposite in 1954 as it was in 1896!

Over time the ways in which we interpret and apply the Constitution change. The Supreme Court adjudicates the interpretations and applications. Some jurists are considered originalists who seek to interpret the Constitution strictly in terms of the framers original intent. Even this strictness cannot begin to compare with that of mathematics. In mathematics logic alone is all that is needed to adjudicate the validity of conclusions.