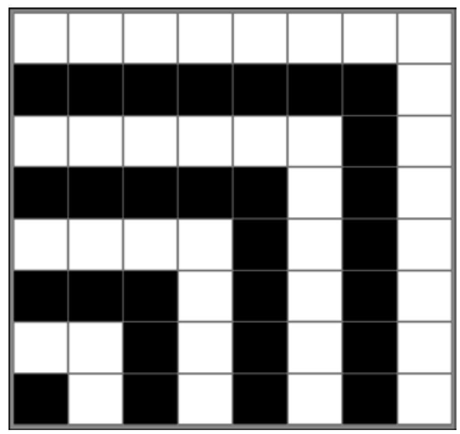
Proofs Without Words

A elegantly executed proof is a poem in all but the form in which it is written.

Morris Kline
(American mathematician and educator; 1908 - 1992)

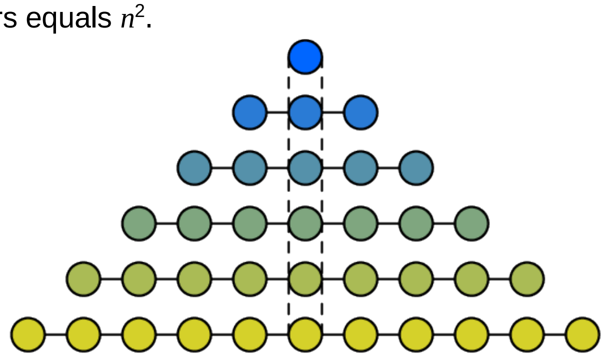
In mathematics, a proof without words is a proof of an identity or mathematical statement which can be demonstrated as self-evident by a diagram without any accompanying explanatory text. Such proofs can be considered more elegant than more formal and mathematically rigorous proofs due to their self-evident nature.[1] When the diagram demonstrates a particular case of a general statement, to be a proof, it must be generalisable.

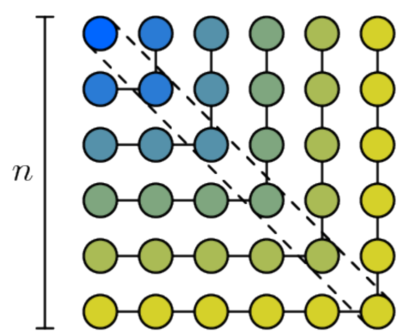


The statement that the sum of all positive odd numbers up to $2n - 1$ is a perfect square - more specifically, the perfect square n^2—can be demonstrated by a proof without words, as shown on the right. The first square is formed by 1 block; 1 is the first square. The next strip, made of white squares, shows how adding 3 more blocks makes another square: four. The next strip, made of black squares, shows how adding 5 more blocks makes the next square.

1. Explain how this process can be continued indefinitely.

2. Explain how the following diagram proves the same theorem: the sum of the first n natural numbers equals n^2.





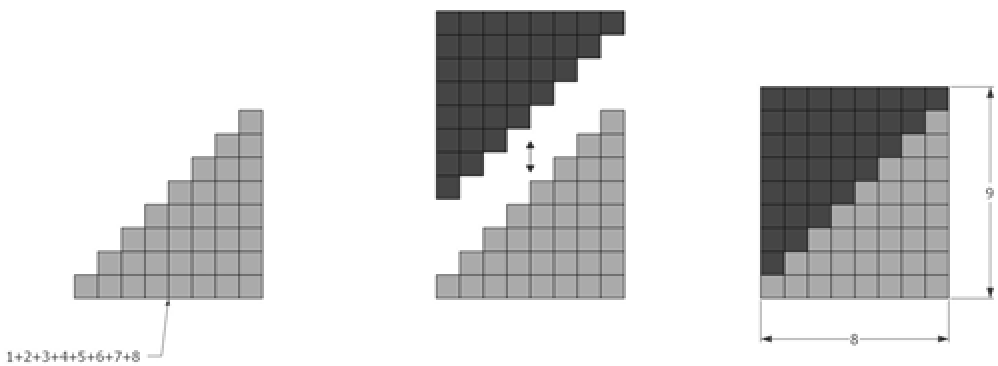
[1] Dunham, William (1974), *The Mathematical Universe*, John Wiley and Sons,p. 120

Mathematical folklore holds that the great **Carl Freidrich Gauss** (German mathematician and scientist; 1777 - 1855) was once, as a very young child, scolded by being sent to the coat closet with a slate to determine the sum of the first hundred numbers: 1 + 2 + 3 + ... + 99 + 100.

The legend holds that he returned within a minute with the correct answer.

The figure below illustrates Gauss's method as it can be represented with blocks to determine the sum

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8.



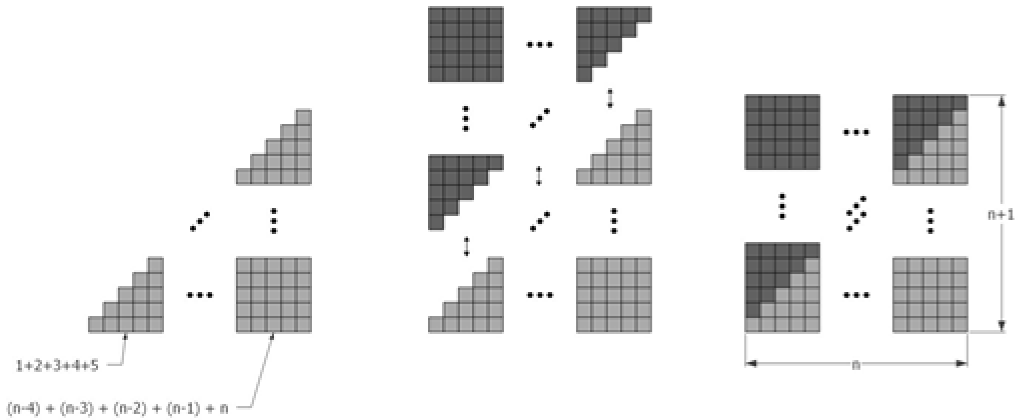
3. Use Gauss's method to determine the sum Gauss was required to compute.

4. Use Gauss's method to determine the sum 1 + 2 + 3 + ... + 1,000,000,000,000.

5. Suppose that n is a positive integer. Find an algebraic expression for the value of the sum $1 + 2 + 3 + \ldots (n-2) + (n-1) + n$.

6. Check that your result agrees with your answers to the two explicit problems computed previously.

7. Explain how the Figure below provides a *proof without words* which proves the general result in #6



Determining the sum 1 + 2 + 3 + ...$(n-2)$ + $(n-1)$ + n

8. Determine the value of the following sums:

1 + 2 + 1 =

1 + 2 + 3 + 2 + 1 =

1 + 2 + 3 + 4 + 3 + 2 + 1 =

1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 =

9. What pattern do you see? Describe this pattern using the language of an algebraic equation.

10. Create a proof without words for this result.

11. Determine the value of the following sums:

1 + 3

1 + 3 + 5

1 + 3 + 5 + 7

1 + 3 + 5 + 7 + 9

12. What pattern do you see? Describe this pattern using the language of an algebraic equation.

13. Create a proof without words for this result.

14. Create proofs without words for the following infinite sums:

a. $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots=\frac{1}{2}$

b. $\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots=\frac{1}{3}$

15. Explain, in detail, the following proof without words.