A Famous Puzzle

Hey, you wanna focus on the problem at hand?

Zeus Carver (Character played by Samuel L. Jackson; -)

The 1995 hit movie *Die Hard with a Vengeance* focuses on a terrorist who has placed bombs around New York City. Detective John McClane, played by **Bruce Willis** (American actor; 1955), is led by the terrorist on a deadly game of Simon Says.

In one scene McClane and Zeus Carver, played by Samuel L. Jackson (American actor and director; 1948 -), are lead to a metal briefcase on the side of a fountain in a crowded city playground. McClane opens the briefcase. A video display monitor reads: “I am a bomb. You have just armed me.”

A phone in the briefcase then rings. The terrorists tells McClane and Carver:

> On the fountain there should be two jugs...a five gallon and a three gallon. Fill one of the jugs with exactly four gallons of water and place it on the scale and the [bomb] timer will stop. You must be precise, one ounce or more/less will result in detonation. If you’re still alive in five minutes…

1. Check the exact time right now. Try to solve this problem in five minutes. If you were unable to solve the problem in five minutes, continue until you can solve it. When you are done, report on how long it took you. More importantly, describe how you worked on this problem, what obstacles you faced, and how you were eventually able to solve the problem.

We’d like to work on some similar problems. So let us make the problem a bit more precise. First, you have an unlimited volume of water at your disposal to use. Second, you have only the two jugs to use. The required amount of water must be exactly contained in one of these jugs.

2. With the same set-up as above, can you fill one of the jugs with exactly 2 gallons? If so, explain how. If not, can you prove why you cannot?
3. With the same set-up as above, can you fill one of the jugs with exactly 1 gallon of water? If so, explain how. If not, can you prove why you cannot?

Simeon Denis Poisson (French mathematician, scientist, and teacher; 1781 - 1840) is one of the greatest French scientists of all time and is also one of the most important mathematicians of the nineteenth century. He was fragile as a young child and had been predeceased by several older brothers and sisters. His father was a soldier who was discriminated against prior to the French Revolution of 1789. After the revolution the elder Poisson had high hopes that his oldest surviving son would rise to a higher social status than his parents.

In *Mathematics and the Imagination*, Edward Kasner (American mathematician; 1878 - 1955) and James Newman (American mathematician; 1907 - 1966) tell us:

> Poisson's family tried to make him everything from a surgeon to a lawyer, that last on the theory that he was fit for nothing better. One or two of these professions he tackled with singular ineptitude.

They continue by telling us how he found his inspiration:

> It was on a journey that someone posed him a problem similar to the one below. Solving it immediately, he realized his true calling and thereafter devoted himself to mathematics.

The problem, Investigation 4 - Investigation 5, is a version of the one we have been considering.

4. An 8 liter jug of wine is to be shared among two friends. In addition to the 8 liter jug, they friends have a 5 liter jug and a 3 liter jug. They would like to share the wine equally without spilling any. How is this problem similar to the "Die Hard" problem? How is it different?

5. Solve this problem. As above, when you are done, report on how long it took you. More importantly, describe how you worked on this problem, what obstacles you faced, and how you were eventually able to solve the problem.

Poisson was not the only famous mathematician to be intrigued by this problem. **Niccol Fontana Tartaglia** (Italian Mathematician and Engineer; 1499 - 1557), also worked on this problem.

Of course, there are many ways we can state problems like this. Let's return to the type of problem posed in *Die Hard with a Vengeance* - unlimited supply of water and two jugs.

6. Suppose you have an 11 liter jug and a 6 liter jug. Can you measure 5 liters?

7. Can you measure 3 liters? How or why not?

8. Can you measure 8 liters? How or why not?

9. Is there any number of liters ($\leqslant 11$) that you cannot measure? Explain in detail.

10. Now suppose you have an 10 liter jug and a 6 liter jug. Can you measure 4 liters?

11. Can you measure 8 liters? How or why not?

12. Can you measure 5 liters? How or why not?

13. Is there any number of liters (≤ 10) that you cannot measure? Explain in detail.

14. Now suppose you have an 15 liter jug and a 6 liter jug. Can you measure 9 liters?

15. Can you measure 12 liters? How or why not?

16. Can you measure 13 liters? How or why not?

17. Is there any number of liters (≤ 15) that you cannot measure? Explain in detail.

18. Do you see a pattern that you can extend to make a conjecture about what units can be measured for a specific choice of jug sizes? Explain precisely.

This problem can be translated into an algebraic problem. Once this is done there is a deep and beautiful connection between this problem and the solution of *Diophantine equations* of the form

$$n \cdot x + m \cdot y = gcd(n, m)$$

where *n* and *m* are the sizes of the jugs.

19. We have put this puzzle here because we believe that each of these puzzles create a simple microworld. Explain why this may be appropriate.

20. For at least one of the situations you have found an amount that you cannot seem to measure, prove rigorously that this amount cannot be measured

21. Have you found a limit to knowledge in this microworld? How firm is this limit? Explain.