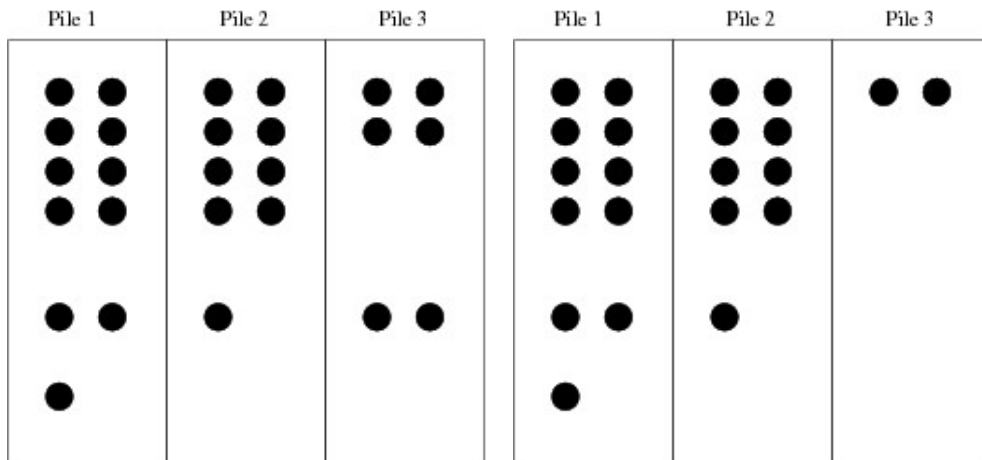# Going Modular

## Analyzing a Game of Nim

The four diagrams below show details of a Three Piles game between Carlos and Amanda. The piles hold 11, 9, and 6 stones, respectively. It's Amanda's turn. Notice how she arranges the stones in the first frame (below left), before making her first move (pictured below right).



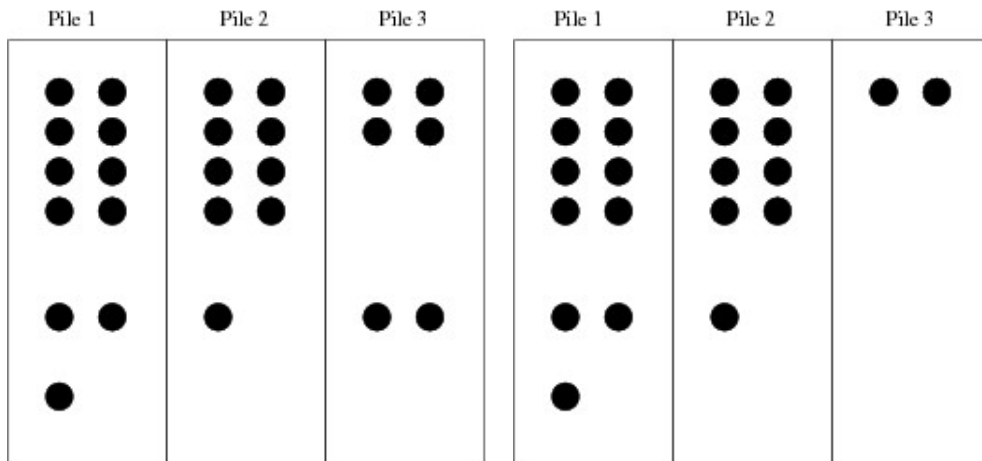(a) Amanda arranges the three piles.    (b) Amanda takes 4 stones from pile three.
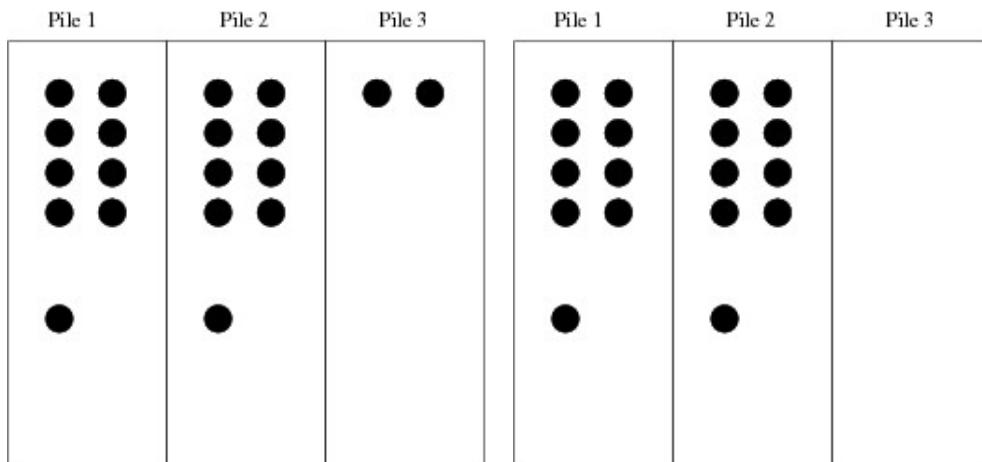


(c) Carlos decides to take 2 stones from pile one.  (d) Amanda decides to take two stones from pile three.

1. What is Amanda doing? Describe any patterns you notice in the arrangements.

2. When asked, Amanda reports: "I pair up sub-piles." Explain what she might mean by that.

3. Who is going to win the game? Explain why.

Pictured below, we see another game. Amanda starts out in the same way, but now Carlos is making different choices.

4. What is Amanda doing? Describe any patterns you notice in the arrangements.

5. Who is going to win the game? Explain why.

6. What can you say about Amanda's strategy of "pairing up sub-piles?" Explain what she might mean by that.



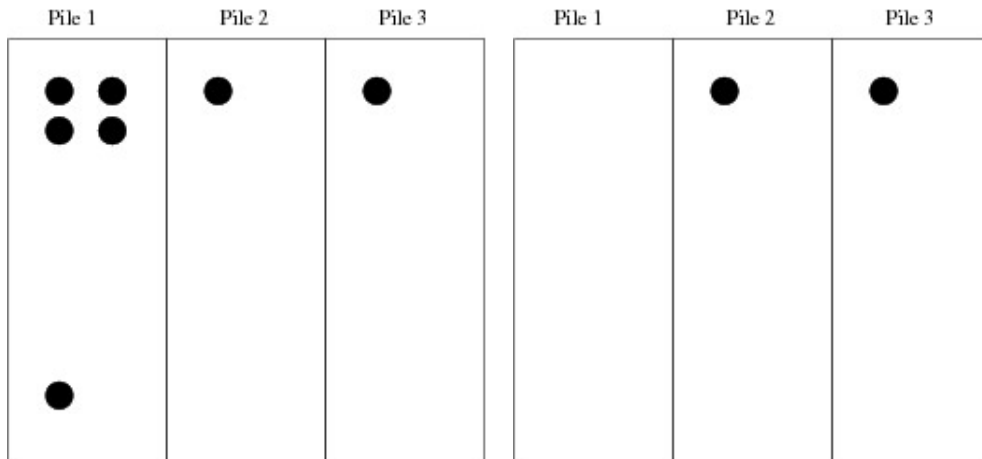(a) Amanda arranges the three piles.  (b) Amanda takes 4 stones from pile three.
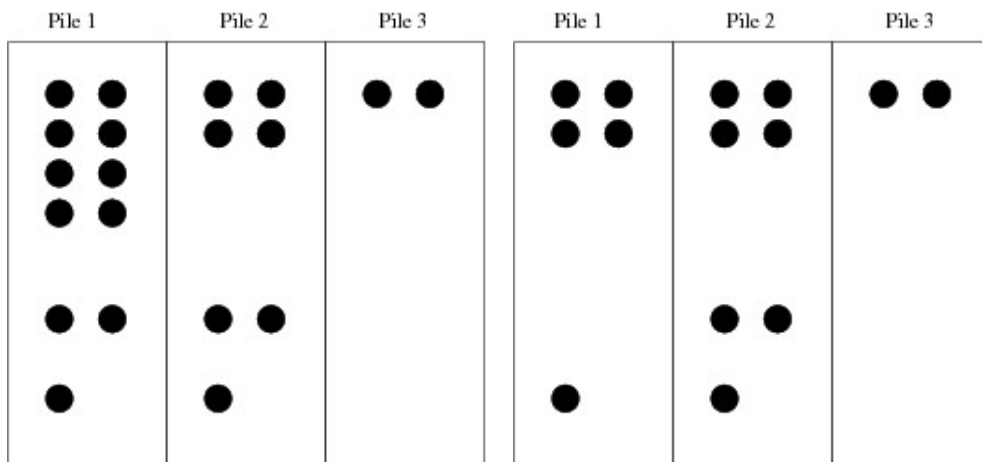


(c) Carlos decides to take 2 stones from pile two. (d) Amanda decides to take 6 stones from pile one. Amanda rearranges the piles as shown.

| Pile 1 | Pile 2 | Pile 3 |
|---|---|---|
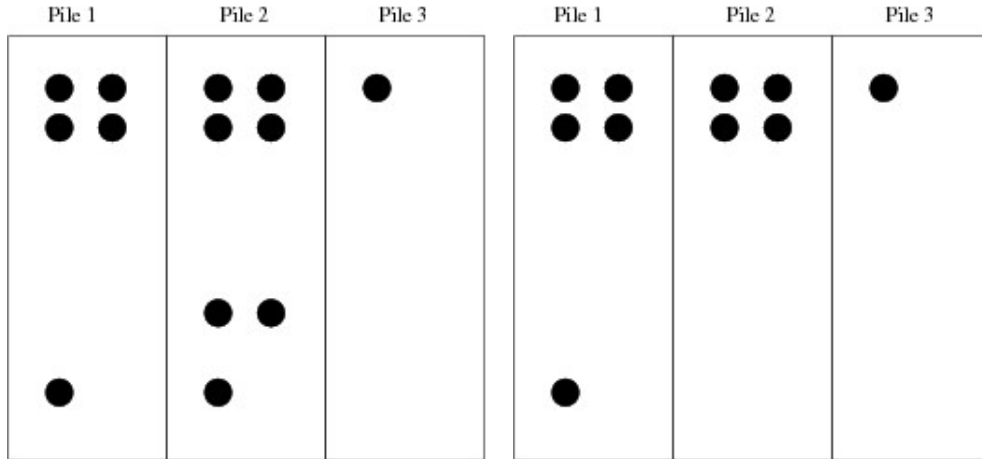
(e) Jim decides to take 1 stone from pile three.

| Pile 1 | Pile 2 | Pile 3 |
|---|---|---|

(f) Amanda decides to take 3 stones from pile two.

| Pile 1 | Pile 2 | Pile 3 |
|---|---|---|

(g) Jim decides to take 3 stones from pile two.

| Pile 1 | Pile 2 | Pile 3 |
|---|---|---|

(h) Amanda decides to take all stones from pile one.

Pictured above: Second game of Three Pile between Carlos and Amanda, continued.

7. With your partner, explore some of the strategies you observed. Play a few games, keep track in your notebook of your moves, stone arrangements, and strategies.

8. Once you feel clear about your strategies, determine your first move for the following arrangements. Explain your strategy.

    a. Three piles with 10, 7, 5 stones, resp.

    b. Three piles with 8, 7, 4 stones, resp.

9. So far we have seen sub-piles of size 1, 2, 4, and 8. How do you think this sequence will continue? Explain.


10. Carlos does not want to draw pictures into his notebook for a game with 7, 5, 2 stones. Instead, he writes 7 = 2 + 2 + 2 + 1, 5 = 4 + 1, 2 = 2 to show his sub-pile arrangement. It is his turn. He is convinced that he will win by taking 4 from the second pile. What do you think is his reasoning?


11. Do you think he is correct? Explain.


12. His opponent, Amanda writes the remaining sub-piles as 7 = 4 + 2 + 1, 1 = 1, 2 = 2. She is convinced that *she* will win, by taking 4 from the first pile. Do you think she is correct? Explain.


13. What important aspect of the winning strategy does this example reveal?


14. Consider the following game position, with piles of size 15, 25, and 12. Use the table to determine the sub-piles, and decide whether you want to go first or second with this game.

| Pile size | 16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|
| 15<br>25<br>12 | 0 | 1 | 1 | 1 | 1 |


15. Now consider the following game position, with piles of size 42, 17, and 57. Use the table to determine the sub-piles, and decide whether you want to go first or second with this game.

| Pile size | 16 | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|
| 42<br>17<br>57 | | | | | |


**16. Summary:** In your own words, describe the "sub-pile strategy."


### 5.3.1   Why does it work?
This brief section summarizes the main ingredients that can allow us to prove why the "subpile" strategy will always work.
Every number can be expressed in a unique way as a binary number.
Express piles sizes as binary numbers.

Add each of the slots using *Nim-sum*: 1 + 1 = 0!
For each of the games you have worked on so far, determine the Nim-sum at each turn. What kind of patterns do you observe?