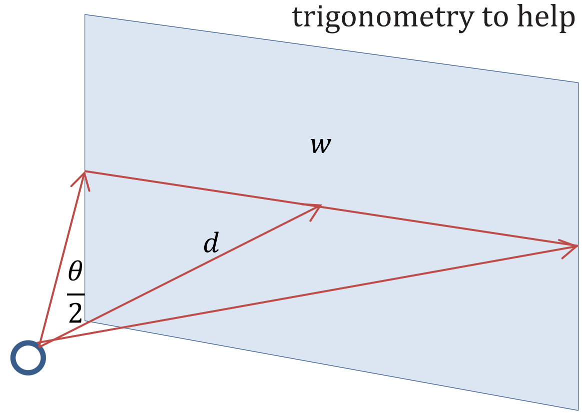
3.6 3-D Projections

Practice Tasks

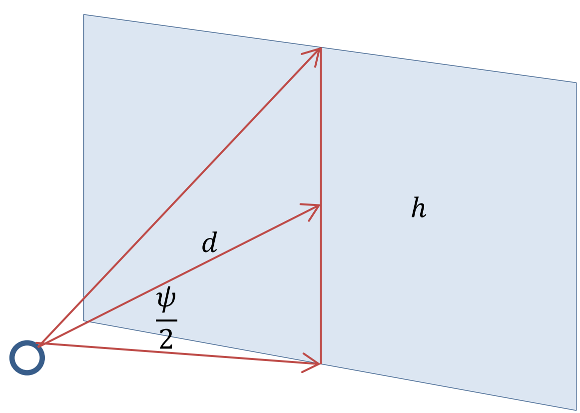
I. Problem Solving

1. Projecting the image of a three-dimensional scene onto a computer screen has the added constraint of the screen size limiting our field of view, or FOV. When we speak of FOV, we wish to know what angle of view the scene represents. Humans have remarkably good peripheral vision. In New York State, the requirement for a driver's license is a horizontal FOV of no less than 140°. There is no restriction placed on the vertical field of vision, but humans normally have a vertical FOV of greater than 120°.

   a. Consider the (simulated) distance the camera is from the screen as d, the horizontal distance of the screen as w, and the horizontal FOV as θ, then use the diagram below and right-triangle trigonometry to help you find θ in terms of w and d.



   b. Repeat procedures from part (a), but this time let h represent the height of the screen and ψ represent the vertical FOV.



c. If a particular game uses an aspect ratio of 16: 9 as its standard view and treats the camera as though it were 8 units away, find the horizontal and vertical FOVs for this game. Round your answers to the nearest degree.

d. When humans sit too close to monitors with FOVs less than what they are used to in real life or in other games, they may grow dizzy and feel sick. Does the game in part (c) run the risk of that? Would you recommend this game be played on a computer or on a television with these FOVs?

2. Computers regularly use polygon meshes to model three-dimensional objects. Most polygon meshes are a collection of triangles that approximate the shape of a three-dimensional object. If we define a *face* of a polygon mesh to be a triangle connecting three vertices of the shape, how many faces at minimum do the following shapes require?

a. A cube.

b. A pyramid with a square base.

c. A tetrahedron.

d. A rectangular prism.

e. A triangular prism.

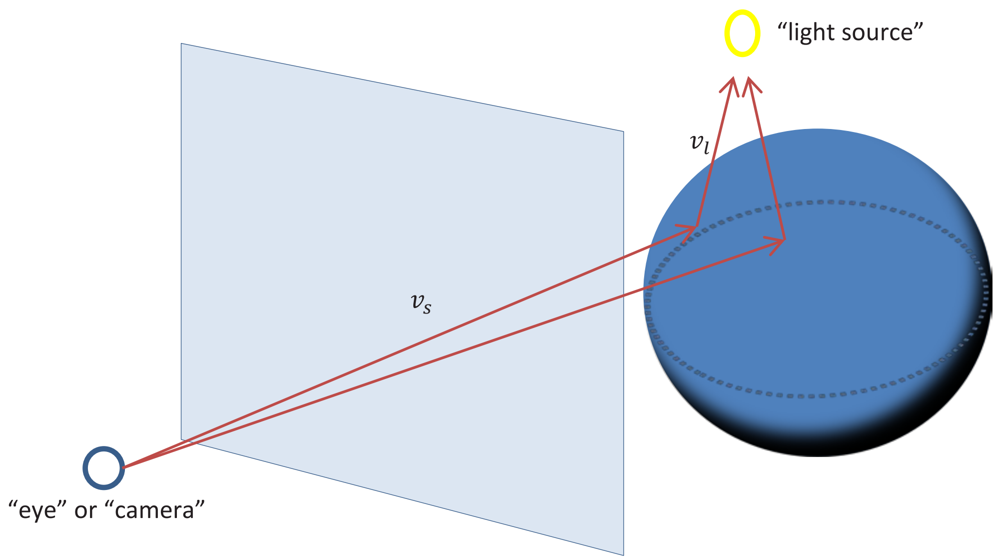
f. An octahedron.

g. A dodecahedron.

h. An icosahedron.

i. How many faces should a sphere have?

3. In the beginning of 3-D graphics, objects were created only using the wireframes from a polygon mesh without shading or textures. As processing capabilities increased, 3-D models became more advanced, and shading and textures were incorporated into 3-D models. One technique that helps viewers visualize how shading works on a 3-D figure is to include both an "eye" and a "light source." Vectors are drawn from the eye to the figure, and then reflected to the light (this technique is called ray tracing). See the diagram below.



a. Using this technique, the hue of the object depends on the sum of the magnitudes of the vectors. Assume the eye in the picture above is located at the origin, v_s is the vector from the eye to the location $\begin{pmatrix}4\\5\\3\end{pmatrix}$, and the light source is located at $\begin{pmatrix}5\\6\\8\end{pmatrix}$. Then find v_l, the vector from v_s to the light source, and the sum of the magnitudes of the vectors.

b. What direction does light travel in real life, and how does this compare to the computer model portrayed above? Can you think of any reason why the computer only traces the path of vectors that start at the "eye"?

III. Reasoning

1. A cube in 3-D space has vertices $\begin{pmatrix}10\\10\\10\end{pmatrix}, \begin{pmatrix}13\\10\\10\end{pmatrix}, \begin{pmatrix}10\\13\\10\end{pmatrix}, \begin{pmatrix}10\\10\\13\end{pmatrix}, \begin{pmatrix}13\\13\\10\end{pmatrix}, \begin{pmatrix}13\\10\\13\end{pmatrix}, \begin{pmatrix}10\\13\\13\end{pmatrix}, \begin{pmatrix}13\\13\\13\end{pmatrix}$.

   a. How do we know that these vertices trace a cube?

   b. What is the volume of the cube?

   c. Let $z = 1$. Find the eight points on the screen that represent the vertices of this cube (some may be obscured).

   d. What do you notice about your result in part (c)?

2. An object in 3-D space has vertices $\begin{pmatrix}1\\5\\0\end{pmatrix}, \begin{pmatrix}0\\6\\0\end{pmatrix}, \begin{pmatrix}0\\5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\0\end{pmatrix}, \begin{pmatrix}0\\4\\0\end{pmatrix}$.

   a. What kind of shape is formed by these vertices?
   b. Let $y = 1$. Find the five points on the screen that represent the vertices of this shape.

3. Consider the shape formed by the vertices given in Problem 2.

   a. Write a transformation matrix that will rotate each point around the y-axis θ degrees.
   b. Project each rotated point onto the plane $y = 1$ if $\theta = 45°$.
   c. Is this the same as rotating the values you obtained in Problem 3 by 45°?

4. In technical drawings, it is frequently important to preserve the scale of the objects being represented. In order to accomplish this, instead of a perspective projection, an orthographic projection is used. The idea behind the orthographic projection is that the points are translated at right angles to the screen (the word stem *ortho-* means *straight* or *right*). To project onto the xy-plane for instance, we can use the matrix $\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}$.

   a. Project the cube in Problem 1 onto the xy-plane by finding the 8 points that correspond to the vertices.
   b. What do you notice about the vertices of the cube after projecting?
   c. What shape is visible on the screen?
   d. Is the area of the shape that is visible on the screen what you expected from the original cube? Explain.
   e. Summarize your findings from parts (a)–(d).
   f. State the orthographic projection matrices for the xz-plane and the yz-plane.
   g. In regard to the dimensions of the orthographic projection matrices, what causes the outputs to be
   two-dimensional?

5. Consider the point $A = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ in the field of view from the origin through the plane $z = 1$.

   a. Find the projection of A onto the plane $z = 1$.
   b. Find a 3×3 matrix P such that PA finds the projection of A onto the plane $z = 1$.
   c. How does the matrix change if instead of projecting onto $z = 1$, we project onto $z = c$, for some real number $c \neq 0$?
   d. Find the scalars that will generate the image of A onto the planes $x = c$ and $y = c$, assuming the image exists. Describe the scalars in words.

Extension:

6. Instead of considering the rotation of a point about an axis, consider the rotation of the camera. Rotations of the camera will cause the screen to rotate along with it, so that to the viewer, the screen appears immobile.

   a. If the camera rotates θ_x around the x-axis, how does the computer world appear to move?
   b. State the rotation matrix we could use on a point A to simulate rotating the camera and computer screen by θ_x about the x-axis but in fact keeping the camera and screen fixed.
   c. If the camera rotates θ_y around the y-axis, how does the computer world appear to move?
   d. State the rotation matrix we could use on a point A to simulate rotating the camera and computer screen by θ_y about the y-axis but in fact keeping the camera and screen fixed.
   e. If the camera rotates θ_z around the z-axis, how does the computer world appear to move?
   f. State the rotation matrix we could use on a point A to simulate rotating the camera and computer screen by θ_z about the z-axis but in fact keeping the camera and screen fixed.
   g. What matrix multiplication could represent the camera starting at a relative angle $(\theta_x, \theta_y, \theta_z)$? Apply the transformations in the order z-y-x. Do not find the product.