8.1: Rational Operations

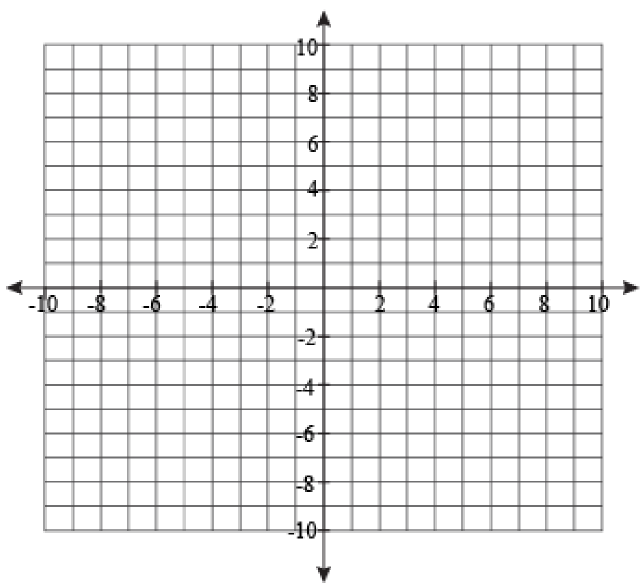
Comparing Rational Expressions to Rational Numbers

A drug is administered to a patient, and the concentration of the drug in the bloodstream is monitored.

At time $t \geq 0$ (in hours since giving the drug), the concentration of the drug (in mg/L) is given by the equation

$$C(t) = \frac{5t}{t^2 + 1}$$

1. Graph the function c with a graphing device. Sketch the graph below.



a. What is the highest concentration of drug that is reached in the patient's bloodstream?

b. At what time does the highest concentration of the drug occur?

c. What happens to the drug concentration after a long period of time?

d. How long does it take for the concentration to drop below 0.3 mg/L?

II. Operations on Rational Expressions

A **rational function** is a function of the form $r(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials

2. Is C (the drug concentration function) a rational function? If it is a rational function identify the functions that correspond to $P(x)$ and $Q(x)$. If it is not a rational expression, explain why not.
3. Perform the following operations on rational numbers and rational expressions:

    a. Add the fractions: $\frac{3}{5} + \frac{2}{7}$.

    b. Subtract the fractions: $\frac{5}{2} - \frac{4}{3}$.

    c. Add the expressions: $\frac{3}{x} + \frac{x}{5}$.

    d. Subtract the expressions: $\frac{x}{x+2} - \frac{3}{x+1}$.

4. Construct an argument that shows that the set of rational numbers is closed under addition. That is, if x and y are rational numbers and $w = x + y$, prove that w must also be a rational number.

5. How could you modify your argument to show that the set of rational numbers is also closed under subtraction? Discuss your response with another student.

6. Perform the following operations on rational numbers and rational expressions:

   a. Multiply the fractions: $\frac{2}{5} \cdot \frac{3}{4}$.

   b. Divide the fractions: $\frac{2}{5} \div \frac{3}{4}$.

   c. Multiply the expressions: $\frac{x+1}{x+2} \cdot \frac{3x}{x-4}$.

   d. Divide the expressions: $\frac{x+1}{x+2} \div \frac{3x}{x-4}$.

7. Construct an argument that shows that the set of rational numbers is closed under division. That is, if x and y are rational numbers (with y nonzero) and $w = \frac{x}{y}$, prove that w must also be a rational number.

8. How could you modify your argument to show that the set of rational expressions is also closed under division by a nonzero rational expression? Discuss your response with another student.

9. How are rational expressions similar to rational numbers?

III. Domain of a Rational Function

The **domain** of a rational function consists of all real numbers x except those for which the denominator is zero. Therefore, whenever you perform algebraic operations on rational expressions, you must exclude the values that make the denominator zero.

10. Find the product or quotient of the two rational expressions. Then identify the excluded values. Part (a) is done as an example.

Product/Quotient	Domain
a. $\frac{1}{x-1} \cdot \frac{x-2}{x+1} = \frac{x-2}{x^2-1}$	All real numbers except $x \neq 1, x \neq -1$
b. $\frac{1}{x-1} \cdot \frac{-1}{x-1} =$	
c. $\frac{x+2}{x-1} \cdot \frac{-x}{x+2} =$	
d. $\frac{x}{x+2} \div \frac{x+1}{x+2} =$	
e. $\frac{x+2}{x} \div \frac{x+1}{x-1} =$	

11. For each pair of functions below, calculate $f(x) + g(x), f(x) - g(x), f(x) \cdot g(x)$ and $\frac{f(x)}{g(x)}$. Indicate restrictions on the domain of the resulting functions.

    a. $f(x) = \frac{2}{x}, g(x) = \frac{x}{x+2}$

    b. $f(x) = \frac{3}{x+1}, g(x) = \frac{x}{x^3+1}$

12. Recall the drug concentration function $C(t) = \frac{5t}{t^2+1}$. Why is the domain all Real numbers – with no excluded values?

IV. Simplifying Rational Expressions

Similar to rational numbers, a rational expression is in **simplified form** when its numerator and denominator have no common factors other than ± 1.

EXAMPLE: Simplify the rational expression $\frac{x^2-4x-12}{x^2-4}$, if possible.

$$\frac{x^2-4x-12}{x^2-4} = \frac{(x+2)(x-6)}{(x+2)(x-2)}$$

$$= \frac{\cancel{(x+2)}(x-6)}{\cancel{(x+2)}(x-2)}$$

$$= \frac{(x-6)}{(x-2)}, \qquad x \neq 2$$

13. Factor each expression completely:

a. $9x^4 - 16x^2$

b. $2x^3 + 5x^2 - 8x - 20$

c. $x^3 + 3x^2 + 3x + 1$

d. $8x^3 - 1$

14. Reduce each rational expression to lowest terms (i.e. simplified form), and specify any excluded values of *x*.

a. $\frac{x^2-6x+5}{x^2-3x-10}$

b. $\frac{x^3+3x^2+3x+1}{x^3+2x^2+x}$

c. $\frac{x^2-16}{x^2+2x-8}$

d. $\frac{x^2-3x-10}{x^3+6x^2+12x+8}$

e. $\frac{x^3+1}{x^2+1}$

f. $\frac{2x^4+6x^3+6x^2+2x}{3x^2+3x}$